MATH1010J University Mathematics, 2019-20

§ 1 Preliminaries

1.1 Notations

Set : collection of objects (elements)

$\subseteq$: subset

$\in$: belongs to

Example 1.1.1

$S = \{1, 2, 3\}$

That means S is a set containing 3 elements, namely 1, 2 and 3.

OR : $1, 2, 3 \in S$

If $T = \{1, 2, 3, 4\}$, then we say S is a subset of T, or $S \subseteq T$.

That means every element in S is also an element in T.

Notations often used in this course :

$\mathbb{Z}^+$: set of all positive integers

$\mathbb{Z}$: set of all integers

$\mathbb{Q}$: set of all rational numbers

$\mathbb{R}$: set of all real numbers

ϕ : empty set, i.e. $\phi = \{ \ \}$ Nothing

$[a,b]$: set of all real numbers x such that $a \leq x \leq b$

(a,b) : set of all real numbers x such that $a < x < b$

$[a,\infty)$: set of all real numbers x such that $a \leq x$

Example 1.1.2

$$\begin{aligned} \text{Set of all positive even integers} &= \{2, 4, 6, \dots\} \\ &= \{2m : m \in \mathbb{Z}^+\} \end{aligned}$$

i.e this set consists of elements of the form $2m$ such that $m \in \mathbb{Z}^+$.

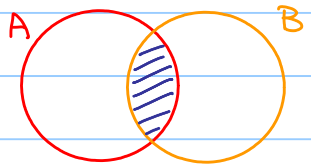
Exercise 1.1.1

Set of all positive odd integers = ? (How to describe ?)

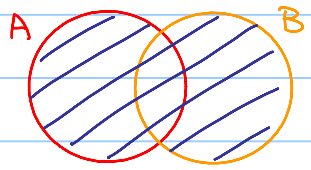
Answer : $\{2m-1 : m \in \mathbb{Z}^+\}$

Set Operations :

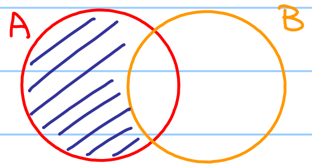
Let A, B be two sets.



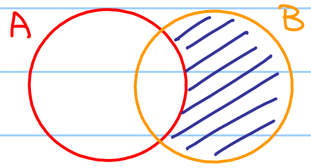
Intersection : $A \cap B$



Union : $A \cup B$



Relative complement of B in A : $A \setminus B$



Relative complement of A in B : $B \setminus A$

Example 1.1.3

Let $A = \{1,2\}$, $B = \{2,3\}$, $C = \{3\}$

- $A \cap B = \{2\}$ $\quad$ $A \cap C = \phi$
- $A \cup B = \{1,2,3\}$
- $A \setminus B = \{1\}$ $\quad$ $B \setminus A = \{3\}$

Example 1.1.4

$\mathbb{R} \setminus \{2\}$: set of all real numbers except 2

(Caution : We cannot write $\mathbb{R} \setminus 2$ as 2 is not a set !)

Example 1.1.5

Solve $x^2 > 1$.

$\therefore$ $x > 1$ or $x < -1$

OR : $x \in (-\infty, -1) \cup (1, \infty)$

OR : $x \in \mathbb{R} \setminus [-1, 1]$

$\forall$: for all

$\exists$: there exists (at least one)

$\exists !$: there exists unique

$\Rightarrow$: implies

$\Leftrightarrow$: if and only if (equivalent to)

s.t. : such that

Example 1.1.6

$\forall\ y \in (0,\infty),\ \exists\ x \in \mathbb{R}$ s.t. $x^2 = y$

$\downarrow$ translate

For all positive real numbers y, there exists (at least one) real number x such that $x^2 = y$.

(In fact, $x = \sqrt{y}$ or $x = -\sqrt{y}$)

$\forall\ y \in (0,\infty),\ \exists !\ x \in (0,\infty)$ s.t. $x^2 = y$

$\downarrow$ translate

For all positive real numbers y, there exists unique positive real number x such that $x^2 = y$.

(In fact, $x = \sqrt{y}$ only!)

Example 1.1.7

Let $x > 0$, $\quad y = \sqrt{x} \overset{\checkmark}{\Rightarrow} y^2 = x$

$\qquad\qquad y^2 = x \overset{\times}{\Rightarrow} y = \sqrt{x}$ (why?)

Example 1.1.8

In $\triangle ABC$,

$\angle ABC = 90^\circ \Rightarrow AB^2 + BC^2 = AC^2$ (Pyth. thm.)

$AB^2 + BC^2 = AC^2 \Rightarrow \angle ABC = 90^\circ$ (Converse of Pyth. thm.)

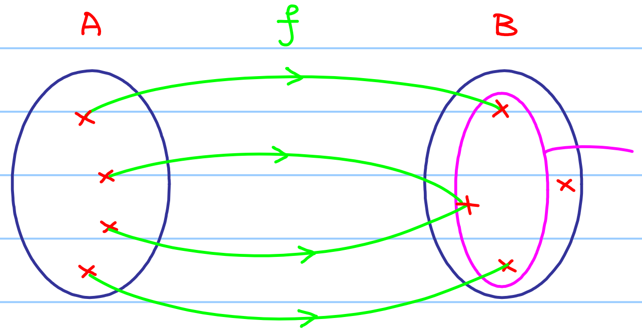
If both statements are true, we say

$\angle ABC = 90^\circ$ if and only if $AB^2 + BC^2 = AC^2$

and we denote it by $\angle ABC = 90^\circ \Leftrightarrow AB^2 + BC^2 = AC^2$

1.2 Functions

Function : A function is a rule that assigns to each element in a set A exactly one element in a set B.



set A : domain (input)

set B : codomain (output)

$\text{range}(f) \subseteq B$: range of f

$$\text{range}(f) = f(A) \underset{\substack{\uparrow\\ \text{defined by}}}{:=} \{ f(x) \in B : x \in A \}$$

A function f from A to B is denoted by $f : A \to B$

Example 1.2.1

If 1) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ $\quad$ $\text{range}(f) = [0, \infty)$

2) $f : [-1, 2) \to \mathbb{R}$ defined by $f(x) = x^2$ $\quad$ $\text{range}(f) = [0, 4)$

Example 1.2.2

If $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 4$

$f(\underset{\substack{\uparrow\\ \text{input}}}{-3}) = (-3)^2 + 4 = \underset{\substack{\uparrow\\ \text{output}}}{13}$ $\quad$ OR write : $\underset{\substack{\uparrow\\ \text{dependent}\\ \text{variable}}}{y} = \underset{\substack{\uparrow\\ \text{independent}\\ \text{variable}}}{x}{}^2 + 4$

Example 1.2.3

If $f(x) = \frac{2x}{x^2 - 7x}$, find the (maximum) domain of f.

Note : $f(x) = \frac{2x}{x^2 - 7x}$ is a well-defined function if $x^2 - 7x \neq 0$.

$$x^2 - 7x = 0$$

$$x(x-7) = 0$$

$$x = 0 \text{ or } 7$$

$\therefore$ Domain of $f = \{ x \in \mathbb{R} : x \neq 0, 7 \}$

$= (-\infty, 0) \cup (0, 7) \cup (7, \infty)$

$= \mathbb{R} \setminus \{0, 7\}$

Example 1.2.4

If $f(x)=\sqrt{x^2-4x+3}$, find the (maximum) domain of f.

Note : $f(x)=\sqrt{x^2-4x+3}$ is a well-defined function if $x^2-4x+3 \geqslant 0$

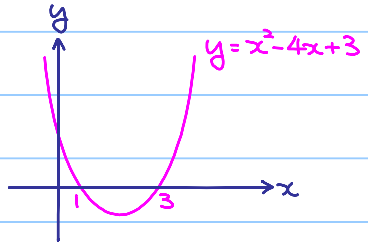
$$x^2-4x+3 \geqslant 0$$

$$x \leqslant 1 \text{ or } x \geqslant 3$$

$\therefore$ Domain of $f = \{x \in \mathbb{R} : x \leqslant 1 \text{ or } x \geqslant 3\}$

$= (-\infty, 1] \cup [3, \infty)$

$= \mathbb{R} \setminus (1,3)$



Exercise 1.2.1

If $f(x)=\dfrac{1}{\sqrt{x^2-4x+3}}$, find the (maximum) domain of f.

Note : $f(x)=\dfrac{1}{\sqrt{x^2-4x+3}}$ is a well-defined function if $x^2-4x+3>0$.

Ans : Domain of $f = \{x \in \mathbb{R} : x<1 \text{ or } x>3\}$

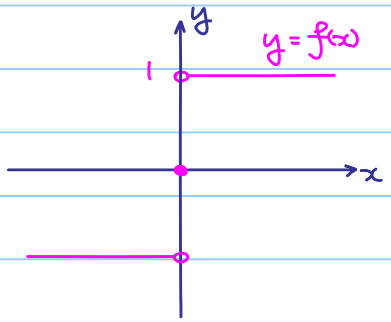
$= (-\infty, 1) \cup (3, \infty)$

$= \mathbb{R} \setminus [1,3]$

Piecewise Defined Function .

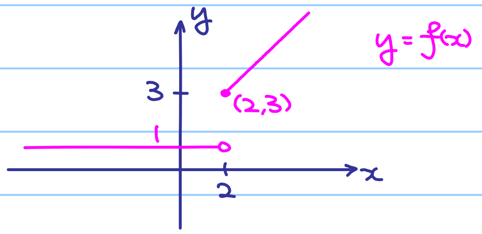
Example 1.2.5

If $f(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$



Example 1.2.6

If $f(x) = \begin{cases} x+1 & \text{if } x \geqslant 2 \\ 1 & \text{if } x<2 \end{cases}$



Exercise 1.2.2

Sketch the graph of $f(x) = \begin{cases} 2x+1 & \text{if } x>1 \\ 0 & \text{if } 0 \leqslant x \leqslant 1 \\ -x^2 & \text{if } x<0 \end{cases}$

Example 1.2.7

Absolute Value : $f(x) = |x| = \sqrt{x^2}$

For example: $|3| = \sqrt{3^2} = \sqrt{9} = 3$

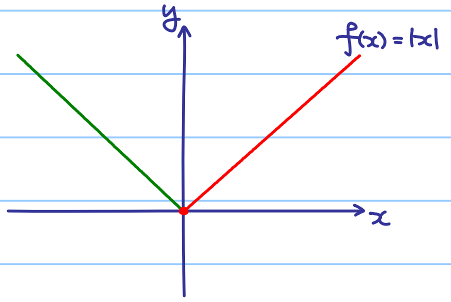
$|0| = \sqrt{0^2} = \sqrt{0} = 0$

$|-3| = \sqrt{(-3)^2} = \sqrt{9} = 3$

(Simply speaking : throw away the negative sign)

Rewrite $|x|$ as a piecewise defined function :

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Example 1.2.8

Let $f(x) = |x+1| + |x-1|$.

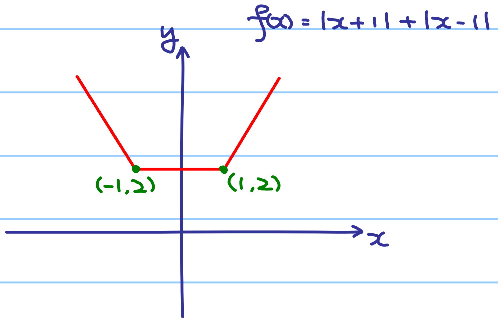
What is the graph of $f(x)$?

Idea : Rewrite $f(x)$ as a piecewise defined function.

Note : $|x+1| = \begin{cases} x+1 & \text{if } x+1 \geq 0 \text{ (i.e } x \geq -1) \\ -(x+1) & \text{if } x+1 < 0 \text{ (i.e. } x < -1) \end{cases}$

$$|x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0 \text{ (i.e. } x \geq 1) \\ -(x-1) & \text{if } x-1 < 0 \text{ (i.e. } x < 1) \end{cases}$$

$$\therefore f(x) = |x+1| + |x-1| = \begin{cases} -(x+1) - (x-1) = -2x & \text{if } x < -1 \\ (x+1) - (x-1) = 2 & \text{if } -1 \leq x < 1 \\ (x+1) + (x-1) = 2x & \text{if } x \geq 1 \end{cases}$$



$f(-1) = 2$ and $f(1) = 2(1) = 2$.

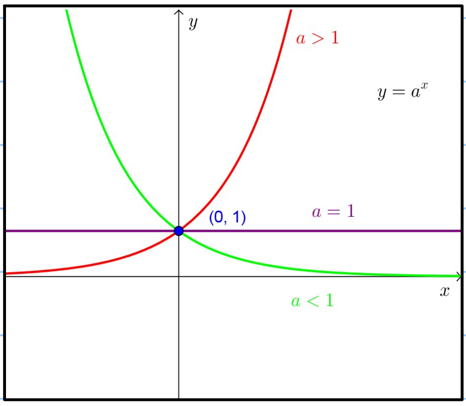
Exponential and Logarithmic Functions :

- $y = a^x$ with $a > 0$

Note : $y = a^x$ is well-defined when $a > 0$!

Think: If $a = -1$, when $x = \frac{1}{2}$, $y = a^x = \sqrt{-1}$!

a^x is positive for any $a > 0$ and any real number x.



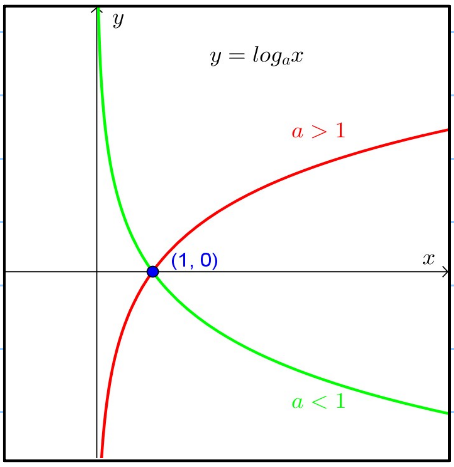
graph of $y = a^x$ for

1) $a > 1$ 2) $a = 1$ 3) $0 < a < 1$

- $y = \log_a x$ with $a > 1$ or $0 < a < 1$

Note : $y = \log_a x$ is well-defined when $a > 1$ or $0 < a < 1$!

By definition, if $y = a^x$, then $\log_a y = x$



graph of $y = \log_a x$ for

1) $a > 1$ 2) $0 < a < 1$

Facts :

1) $\log_a M + \log_a N = \log_a MN$

2) $\log_a M - \log_a N = \log_a \frac{M}{N}$

3) $\log_a M^n = n \log_a M$

4) $\log_a x = \frac{\log_b x}{\log_b a}$ (Change of base)

5) $e = 2.71828\ldots$ (Explain later)

We write $\log_e x$ as $\ln x$ (natural log function)

6) a^x and $\log_a x$ are inverse to each other,

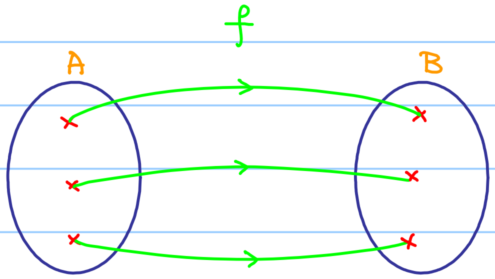
i.e. $a^{\log_a x} = x$ and $\log_a a^x = x$

Injective and Surjective Functions

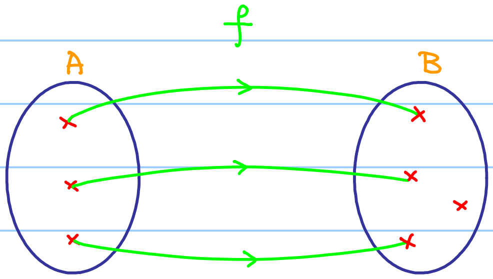
Intuitive idea:

injective : every $y \in \text{range}(f)$ comes from exactly one $x \in A$

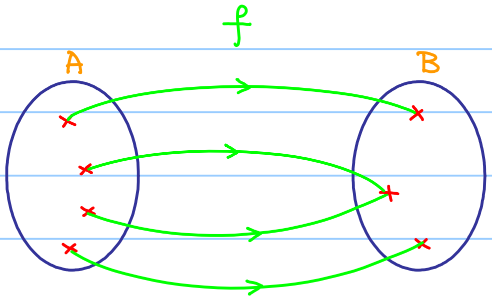
surjective : every $y \in B$ comes from at least one $x \in A$



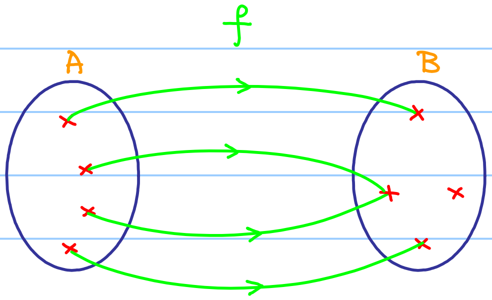
Both injective and surjective



injective but not surjective



surjective but not injective



Neither injective nor surjective

Definition 1.2 1

Let $f: A \to B$ be a function.

1) f is said to be an injective function if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

(Explanation : Once the output are the same , the inputs must be the same !)

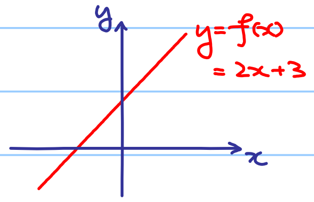
2) f is said to be a surjective function if

$$\forall y \in B, \exists x \in A \text{ s.t. } f(x) = y \quad (\text{i.e } f(A) = B)$$

If f is both injective and surjective , then it is said to be bijective .

Example 1.2.9

Show that $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x+3$ is a bijective function.



1) Injective :

$$f(x_1) = f(x_2)$$

$$\Rightarrow \quad 2x_1 + 3 = 2x_2 + 3$$

$$\Rightarrow \quad x_1 = x_2$$

$\therefore$ f is injective.

2) Surjective :

Let $y \in \mathbb{R}$,

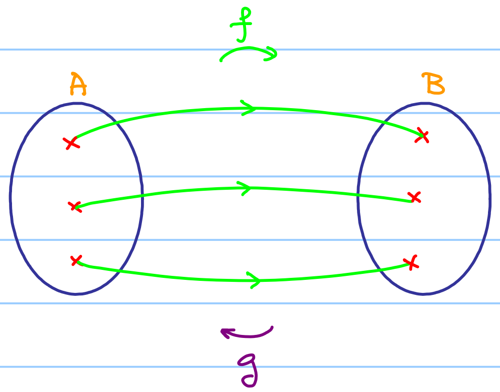
take $x = \frac{y-3}{2} \in \mathbb{R}$

$$f(x) = f\left(\frac{y-3}{2}\right)$$
$$= 2\left(\frac{y-3}{2}\right) + 3$$
$$= y$$

$\therefore$ f is surjective.

Inverse of a Function :

Intuitive idea :



Definition 1.2.2

Let $f: A \to B$ be a function. If $g \cdot B \to A$ is a function such that

1) $g(f(x)) = x \quad \forall x \in A$

2) $f(g(y)) = y \quad \forall y \in B$

Then g is said to be an inverse of f.

Fact : 1) Once an inverse of f exists, it is unique, so we denote it by f^{-1}

2) f has an inverse if and only if f is bijective.

Example 1.2.10

		injective	surjective
$f:\mathbb{R}\to\mathbb{R}$	defined by $f(x) = \sin x$	✗	✗
$f:\mathbb{R}\to[-1,1]$	defined by $f(x) = \sin x$	✗	✓
$f:[-\frac{\pi}{2},\frac{\pi}{2}]\to[-1,1]$	defined by $f(x) = \sin x$	✓	✓

$f:[-\frac{\pi}{2},\frac{\pi}{2}]\to[-1,1]$ defined by $f(x) = \sin x$ is a bijective function

$\therefore$ we can define arcsin function !

$\sin^{-1}: [-1,1]\to[-\frac{\pi}{2},\frac{\pi}{2}]$

We write $\sin : [-\frac{\pi}{2},\frac{\pi}{2}]\to[-1,1]$

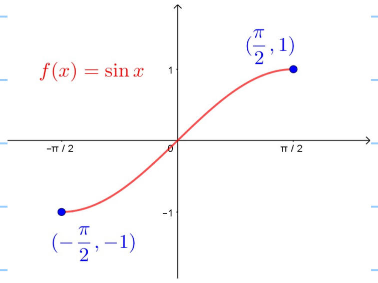
$\sin^{-1}(\sin x) = x \quad \forall\ x\in[-\frac{\pi}{2},\frac{\pi}{2}]$

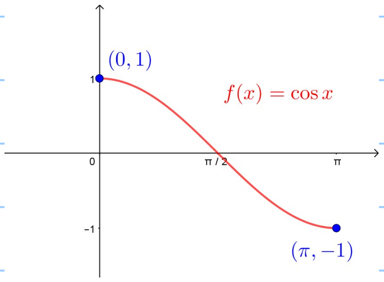
$\sin(\sin^{-1}y) = y \quad \forall\ y\in[-1,1]$

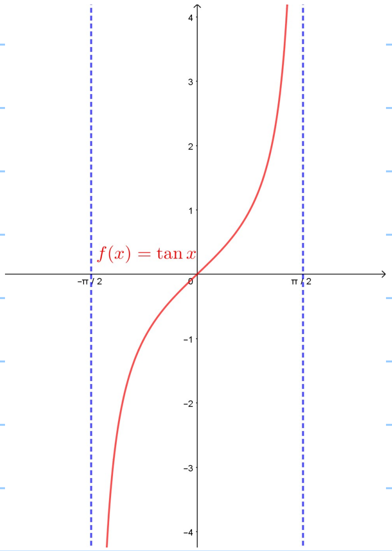
Furthermore,

$\cos : [0,\pi]\to[-1,1]$ is bijective and $\cos^{-1}: [-1,1]\to[0,\pi]$

$\tan : (-\frac{\pi}{2},\frac{\pi}{2})\to\mathbb{R}$ is bijective and $\tan^{-1}: \mathbb{R}\to(-\frac{\pi}{2},\frac{\pi}{2})$







Example 1.2.12

Let $f:\mathbb{R}\to(0,\infty)$ defined by $f(x) = e^x$.

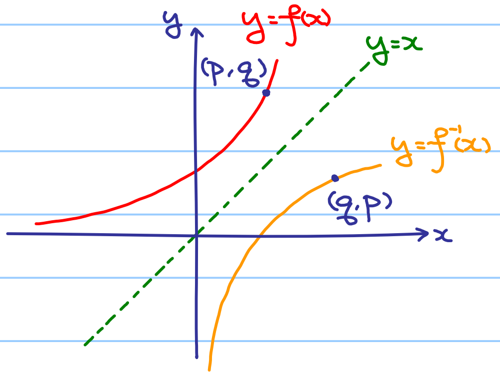
Note that f is bijective, so $f^{-1}: (0,\infty)\to\mathbb{R}$ can be defined and it is denoted by $f^{-1}(x) = \ln x$.

Then, we have

1) $f^{-1}(f(x)) = \ln(e^x) = x \quad \forall x\in\mathbb{R}$;

2) $f(f^{-1}(y)) = e^{\ln y} = y \quad \forall y\in(0,\infty)$.

Fact : The graph of f^{-1} is the reflection of the graph of f along $y=x$.



Reason : (p,q) lies on the graph of f

$\Leftrightarrow f(p)=q$

$\Leftrightarrow f^{-1}(q)=p$

$\Leftrightarrow (q,p)$ lies on the graph of f^{-1}

Even, Odd and Periodic Functions

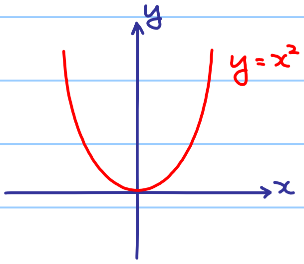
Definition 1.2.3

A function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is said to be

- even if $f(-x)=f(x)$ for all $x\in\mathbb{R}$

  e.g. x^2, $\cos x$, $|x|$

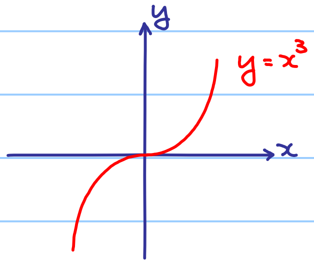
  property : the graph is symmetric along y-axis.



- odd if $f(-x)=-f(x)$ for all $x\in\mathbb{R}$

  e.g. x^3, $\sin x$

  property : the graph is symmetric about the origin



Exercise 1.2.3

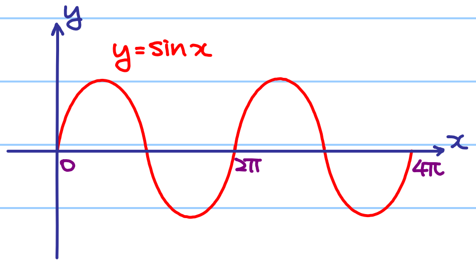
If $f:\mathbb{R}\rightarrow\mathbb{R}$ is an odd function, show that $f(0)=0$.

- periodic if there exists $T>0$ such that $f(x)=f(x+T)$ for all $x\in\mathbb{R}$

  If $T>0$ is the least positive real number with the above property, T is called the period.

  e.g. $\sin x$, $\cos x$, $\tan x$

  property : the graph is repeating again and again



period of $\sin x$, $\cos x$ $=2\pi$

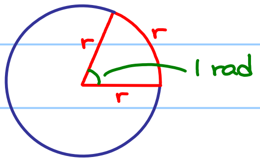
period of $\tan x$ $=\pi$

1.3 Trigonometry

Another unit of measurement of angles (radian) :

Definition 1.3.1

When the length of an arc equals to the radius, the angle suspended is defined as 1 radian.



Direct consequence . 2π rad $= 360°$

Exercise : π rad $=$ _____

_______ $= 90°$

_______ $= 60°$

Remark : From now on, use radian .

Area of sector $A \propto \theta$

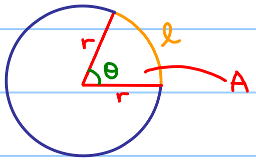
$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$A = \frac{1}{2}r^2\theta$$

Arclength $\ell \propto \theta$

$$\frac{\ell}{2\pi r} = \frac{\theta}{2\pi}$$

$$\ell = r\theta$$



Trigonometric identities :

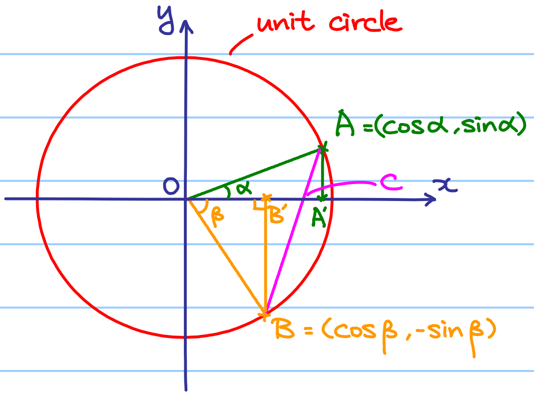
① Consider the length of AB :

i) $AB^2 = OA^2 + OB^2 - 2\cos(\alpha+\beta) = 2 - 2\cos(\alpha+\beta)$

ii) $AB^2 = (AA' + BB')^2 + (A'B')^2 = (\sin\alpha + \sin\beta)^2 + (\cos\alpha - \cos\beta)^2$

$= 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$

$\therefore \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$



② Join AB , AB cuts the x-axis at C. Then $C = \left(\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha + \sin\beta}, 0\right)$

Consider the area of ΔOAB :

i) area of $\Delta OAB = \frac{1}{2} OA \cdot OB \cdot \sin(\alpha+\beta) = \frac{1}{2}\sin(\alpha+\beta)$

ii) area of ΔOAB = area of ΔOAC + area of ΔOBC

$= \frac{1}{2} \cdot OC \cdot AA' + \frac{1}{2} \cdot OC \cdot BB'$

$= \frac{1}{2} \cdot OC \cdot (AA' + BB')$

$= \frac{1}{2} \cdot \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha + \sin\beta} (\sin\alpha + \sin\beta) = \frac{1}{2} \cdot (\sin\alpha\cos\beta + \cos\alpha\sin\beta)$

$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Theorem 1.3.1

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Compound angle formula

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

) replace β by $-\beta$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

) replace β by $-\beta$

↓ taking quotient of the 1st and the 3rd eq^n

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

Double angle formula

$$\xrightarrow{\text{put } \beta=\alpha} \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$= 2\cos^2\alpha - 1 = 1-2\sin^2\alpha$$

$$\xrightarrow{\text{put } \beta=\alpha} \sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\xrightarrow{\text{put } \beta=\alpha} \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Product to sum formula

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

$$-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$$

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

put $\alpha = \frac{A+B}{2}$, $\beta = \frac{A-B}{2}$

Sum to product formula

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

1.4 Sequences of Real Numbers

Example 1.4.1

Let $a_1 = 2$, $a_2 = \pi$, $a_3 = \sqrt{3}$, $\cdots$

OR write as $\{2, \pi, \sqrt{3}, \cdots\}$ (No pattern)

Example 1.4.2

Sequences having patterns :

Let $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $\cdots$ in general , $a_n = 2^{n-1}$

Let $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $\cdots$ in general , $a_n = \frac{1}{n}$

Let $a_1 = -1$, $a_2 = 1$, $a_3 = -1$, $\cdots$ in general , $a_n = (-1)^n$

Example 1.4.3

Recursive sequence.

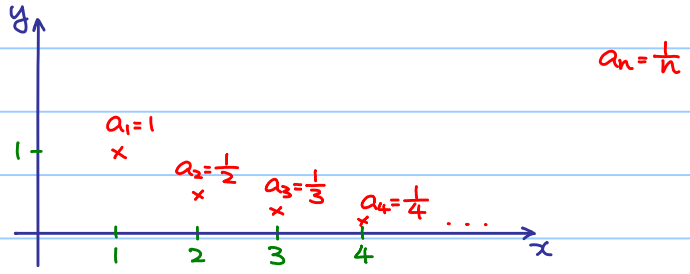
Let $\{a_n\}$ be a sequence of real numbers defined by $a_1 = 1$ and $a_{n+1} = a_n^2 + 2$ for $n \geq 1$.

Then $\{a_n\} = \{1, 3, 11, 123, \cdot \cdot\}$.

Remark / Definition 1.4.1

A sequence of real numbers $\{a_n\}$ can be regarded as a function $f: \mathbb{Z}^+ \to \mathbb{R}$ and $a_n = f(n)$ (i.e. given $n \in \mathbb{Z}^+$, return the n-th term of the sequence.)

A sequence can be visualized by the following diagram :



Any observation ?

When n is getting larger and larger , a_n is getting closer and closer to 0 .

§ 2 Limits of Sequences

2.1 Definition

Definition 2.1.1 (Informal)

Let $\{a_n\}$ be a sequence of real numbers.

If n is getting larger and larger, a_n is getting closer and closer to $L \in \mathbb{R}$,

then we say L is the limit of the sequence $\{a_n\}$ and we denote it by $\lim\limits_{n\to\infty} a_n = L$.

In this case, $\{a_n\}$ is said to be convergent, or $\{a_n\}$ converges to L.

Example 2.1.1

$\lim\limits_{n\to\infty} \frac{1}{n} = 0$. (However, $\frac{1}{n} \neq 0 \ \forall n \in \mathbb{Z}^+$)

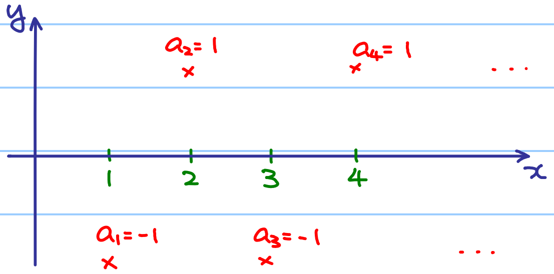
$\lim\limits_{n\to\infty} 2^{n-1}$ does NOT exist.

(But some still write $\lim\limits_{n\to\infty} 2^{n-1} = +\infty$ or

say 2^{n-1} diverges to $+\infty$)

$\lim\limits_{n\to\infty} (-1)^n$ does NOT exist.

$a_n = (-1)^n$

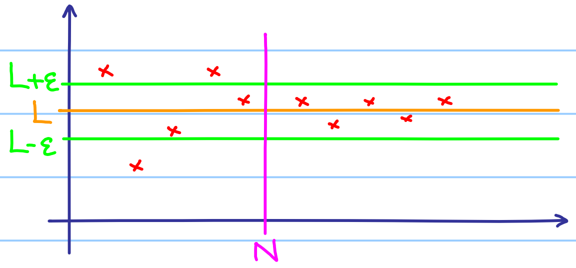


Definition 2.1.2 (ε-definition)

Let $\{a_n\}$ be a sequence of real numbers and $L \in \mathbb{R}$.

L is said to be the limit of the sequence $\{a_n\}$ if

$\forall \varepsilon > 0$, $\exists N \in \mathbb{Z}^+$ s.t. $|a_n - L| < \varepsilon \quad \forall n \geq N$.



Meaning: No matter how small ε is given,

we can always find a $N \in \mathbb{Z}^+$ such that the tail (a_n with $n \geq N$) of the sequence

lies in the ε-tunnel (ε-neighborhood of L)

Theorem 2.1.1

1) If $a_n = k$ $\forall n \in \mathbb{Z}^+$ (constant sequence), then $\lim_{n\to\infty} a_n = k$.

2) If $k>0$ and $a_n = n^{-k} = \frac{1}{n^k}$, then $\lim_{n\to\infty} a_n = 0$.

3) If $-1<a<1$, then $\lim_{n\to\infty} a^n = 0$.

Remark: It seems that all the above are obvious, but we need to check the ε-definition, which is hard.

2.2 Algebraic Properties of Limits

Theorem 2.2.1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ (very important assumption), then

1) $\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L+M$

2) $\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L-M$

3) $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = LM$

4) If $M \neq 0$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}$

Example 2.2.1

Find $\lim_{n\to\infty} \frac{2}{n} + 3$

Logically:

① $\lim_{n\to\infty} 2 = 2$, $\lim_{n\to\infty} \frac{1}{n} = 0$, so $\lim_{n\to\infty} \frac{2}{n} \overset{\text{By (3)}}{=} (\lim_{n\to\infty} 2)(\lim_{n\to\infty} \frac{1}{n}) = 2 \cdot 0 = 0$

② $\lim_{n\to\infty} \frac{2}{n} = 0$, $\lim_{n\to\infty} 3 = 3$, so $\lim_{n\to\infty} \frac{2}{n} + 3 \overset{\text{By (1)}}{=} \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} 3 = 0+3 = 3$

But what we write:

$$\lim_{n\to\infty} \frac{2}{n} + 3 = \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} 3$$
$$= 0+3$$
$$= 3$$

Example 2.2.2

Find $\lim\limits_{n\to\infty} \frac{n^2+3}{2n^2-4n}$

$\lim\limits_{n\to\infty} \frac{n^2+3}{2n^2-4n}$ (We cannot use (4), why ?)

$= \lim\limits_{n\to\infty} \frac{1+\frac{3}{n^2}}{2-\frac{4}{n}}$ (Now, we can use (4) !)

$= \frac{\lim\limits_{n\to\infty} 1+\frac{3}{n^2}}{\lim\limits_{n\to\infty} 2-\frac{4}{n}}$

$= \frac{1}{2}$

Exercise 2.2.1

Find $\lim\limits_{n\to\infty} \frac{3n+1}{n^2-2n}$, $\lim\limits_{n\to\infty} \frac{n^3+2n}{2n^2+1}$ (if exist)

Answer : $\lim\limits_{n\to\infty} \frac{3n+1}{n^2-2n}$ ← grows faster $\lim\limits_{n\to\infty} \frac{n^3+2n}{2n^2+1}$ ← grows faster

$\lim\limits_{n\to\infty} \frac{3n+1}{n^2-2n} = 0$, $\lim\limits_{n\to\infty} \frac{n^3+2n}{2n^2+1}$ does NOT exist

Any observation ?

Basically, we are comparing the degrees of the numerator and the denominator.

Conclusion :

If $p(x)$ and $q(x)$ are polynomials,

$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$ with $a_m \neq 0$ ($\deg p(x) = m$)

$q(x) = b_k x^k + a_{k-1} x^{k-1} + \cdots + b_1 x + b_0$ with $b_k \neq 0$ ($\deg q(x) = k$)

then

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \begin{cases} \pm\infty & \text{if } m > k \\ \frac{a_m}{b_m} & \text{if } m = k \\ 0 & \text{if } m < k \end{cases}$$

Following this idea:

Example 2.2.3

Find $\lim\limits_{n\to\infty} \frac{3n-1}{\sqrt{4n^2+2n}}$

$\lim\limits_{n\to\infty} \frac{3n-1}{\sqrt{4n^2+2n}}$ ⟵ roughly deg = 1

$= \lim\limits_{n\to\infty} \frac{3-\frac{1}{n}}{\sqrt{4+\frac{2}{n}}}$

$= \frac{3}{2}$

Example 2.2.4

Find $\lim\limits_{n\to\infty} \sqrt{n+1}-\sqrt{n}$ (Never say $\lim\limits_{n\to\infty} \sqrt{n+1}-\sqrt{n} = \infty - \infty = 0$)

$\lim\limits_{n\to\infty} \sqrt{n+1}-\sqrt{n}$

$= \lim\limits_{n\to\infty} (\sqrt{n+1}-\sqrt{n}) \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

$= \lim\limits_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$= 0$

Example 2.2.5

Find $\lim\limits_{n\to\infty} \frac{2^n}{n}$.

Question: Can we say $\frac{2^n}{n} = \frac{1}{n}\cdot 2^n$ and $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, so $\lim\limits_{n\to\infty} \frac{2^n}{n} = 0$?

Absolutely NOT!

Since $\lim\limits_{n\to\infty} 2^n$ does NOT exist, property (3) cannot be applied!

2.3 Monotonic Sequence Theorem

Definition 2.3.1

Let $\{a_n\}$ be a sequence of real numbers.

(i) $\{a_n\}$ is said to be bounded above if $\exists M \in \mathbb{R}$ s.t. $a_n \leq M$ (called an upper bound) $\forall n \in \mathbb{Z}^+$

(ii) $\{a_n\}$ is said to be bounded below if $\exists M \in \mathbb{R}$ s.t. $a_n \geq M$ (called an lower bound) $\forall n \in \mathbb{Z}^+$

(iii) $\{a_n\}$ is said to be bounded if $\exists M > 0$ s.t. $|a_n| \leq M$ (i.e. $-M \leq a_n \leq M$) $\forall n \in \mathbb{Z}^+$

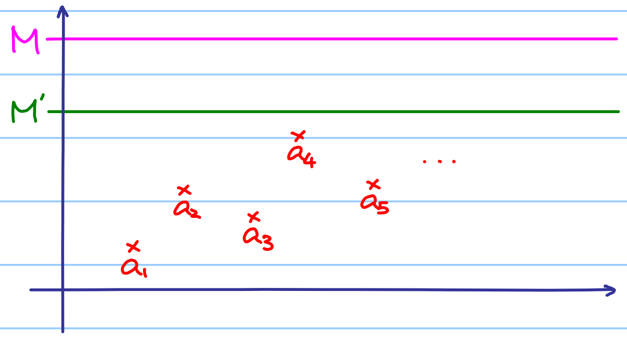
bounded = both bounded above and below

(iv) $\{a_n\}$ is said to be monotonic increasing if $a_{n+1} \geq a_n$ $\forall n \in \mathbb{Z}^+$

(v) $\{a_n\}$ is said to be monotonic decreasing if $a_{n+1} \leq a_n$ $\forall n \in \mathbb{Z}^+$

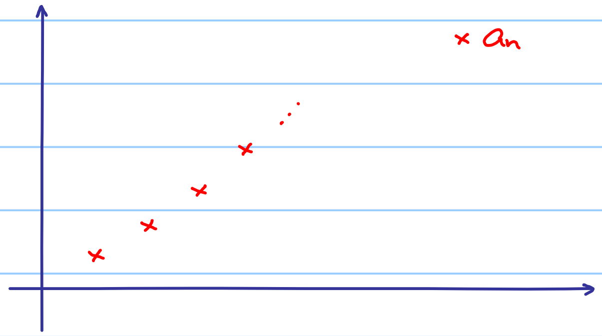
(vi) $\{a_n\}$ is said to be monotonic if it is either monotonic increasing or decreasing

Geometrical meaning :



$\{a_n\}$ is bounded above by M

However, it may happen that a sharper upper bound M′ exists.



$\{a_n\}$ is monotonic increasing.

Example 2.3.1

$\lim_{n\to\infty} 2^{n-1}$ does NOT exist. (monotonic but not bounded) monotonic $\overset{\times}{\Rightarrow}$ convergent

$\lim_{n\to\infty} (-1)^n$ does NOT exist. (bounded but not monotonic) bounded $\overset{\times}{\Rightarrow}$ convergent

How about combining them together ?

Theorem 2.3.1 (Monotone Convergence Theorem)

If $\{a_n\}$ is bounded above (below) and monotonic increasing (decreasing), then $\lim_{n\to\infty} a_n$ exists.

Geometrical meaning:

Caution:

$\{a_n\}$ is bounded above by M,

but $\lim_{n\to\infty} a_n$ is NOT necessary to be M.

Example 2.3.2

Let $\{a_n\}$ be a sequence of positive real numbers defined by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = 1 + \frac{a_n}{1+a_n} \quad (n \geq 1)$$

Does $\lim_{n\to\infty} a_n$ exist?

i) Claim: $\{a_n\}$ is monotonic increasing

Prove the statement "$a_{n+1} \geq a_n$" by induction:

Step 1: $a_2 - a_1 = (1 + \frac{a_1}{1+a_1}) - a_1 = \frac{3}{2} - 1 = \frac{1}{2} > 0$

Step 2: Assume $a_{k+1} \geq a_k$ for some $k \in \mathbb{Z}^+$

$$\begin{aligned} a_{k+2} - a_{k+1} &= (1 + \frac{a_{k+1}}{1+a_{k+1}}) - (1 + \frac{a_k}{1+a_k}) \\ &= \frac{a_{k+1}}{1+a_{k+1}} - \frac{a_k}{1+a_k} \\ &= \frac{a_{k+1} - a_k}{(1+a_{k+1})(1+a_k)} \geq 0 \end{aligned}$$

ii) $\{a_n\}$ is bounded above by 2.

$\therefore$ By Monotone Convergence Theorem, $\lim\limits_{n\to\infty} a_n$ exists (But, what is the value ?)

Let $\lim\limits_{n\to\infty} a_n = A$

Note that $a_{n+1} = 1 + \frac{a_n}{1+a_n}$, taking limit on both sides

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 1 + \frac{a_n}{1+a_n} = 1 + \frac{\lim\limits_{n\to\infty} a_n}{1+\lim\limits_{n\to\infty} a_n}$$

$$A = 1 + \frac{A}{1+A}$$

$$A^2 - A - 1 = 0$$

$$A = \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2} \text{ (rejected)}$$

Note : the limit is NOT 2.

Constant e :

Consider a number $(1+\frac{1}{m})^n$ which depends on both m and n and then

1) fix m, say m = 100, n is getting larger and larger.

$n = 10$	$n = 100$	$n = 1000$	$n \to \infty$
$(1+\frac{1}{m})^n = 1.01^{10}$	$(1+\frac{1}{m})^n = 1.01^{100}$	$(1+\frac{1}{m})^n = 1.01^{1000}$	$(1+\frac{1}{m})^n \to \infty$

2) fix n, say n = 100, m is getting larger and larger.

$m = 10$	$m = 100$	$m = 1000$	$m \to \infty$
$(1+\frac{1}{m})^n = 1.1^{100}$	$(1+\frac{1}{m})^n = 1.01^{100}$	$(1+\frac{1}{m})^n = 1.001^{100}$	$(1+\frac{1}{m})^n \to 1$

How about setting $m = n$ and let them become larger and larger ?

$\lim\limits_{n\to\infty} (1+\frac{1}{n})^n$ exists ? something between 1 and ∞

$n = 10$	$n = 100$	$n = 1000$	$n \to \infty$
$(1+\frac{1}{n})^n = 1.1^{10}$	$(1+\frac{1}{n})^n = 1.01^{100}$	$(1+\frac{1}{m})^n = 1.001^{1000}$	$(1+\frac{1}{n})^n \to 2.71828\ldots$
≈ 2.59374	≈ 2.70481	≈ 2.71692	limit exists and call it e.

Example 2.3.3

Let $\{a_n\}$ be a sequence of real numbers defined by $a_n = (1+\frac{1}{n})^n$.

Prove that

1) $\{a_n\}$ is monotonic increasing;

2) $\{a_n\}$ is bounded above.

$$a_n = (1+\frac{1}{n})^n$$

$$= \sum_{r=0}^{n} C_r^n \frac{1}{n^r}$$ (Totally , n+1 terms)

$$= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \frac{1}{n^3} + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!} \cdot \frac{1}{n^r} + \cdots + \frac{n(n-1)\cdots(n-n+1)}{n!} \cdot \frac{1}{n^n}$$

$$= 1+1+\frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots + \frac{1}{r!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{r-1}{n}) + \cdot\cdot + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})$$

Similarly,

$$a_{n+1} = 1+1+\frac{1}{2!}(1-\frac{1}{n+1}) + \frac{1}{3!}(1-\frac{1}{n+1})(1-\frac{2}{n+1}) + \cdots + \frac{1}{r!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{r}{n+1}) + \cdots + \frac{1}{n!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{n-1}{n+1})$$

$$+ \frac{1}{(n+1)!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{n-1}{n+1})(1-\frac{n}{n+1})$$ (Totally , n+2 terms)

Clearly , $a_{n+1} \geq a_n$, i.e. $\{a_n\}$ is monotonic increasing ;

$$a_n = 1+1+\frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots + \frac{1}{r!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{r-1}{n}) + \cdots + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})$$

$$\leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdot\cdot+\frac{1}{r!}+\cdots+\frac{1}{n!}$$

$$\leq 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdot \quad +\frac{1}{2^{r-1}}+\cdot \quad +\frac{1}{2^{n-1}}$$

$$= 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}$$

$$\leq 1 + \frac{1}{1-\frac{1}{2}}$$

$$= 3$$

$\therefore$ $\{a_n\}$ is bounded above by 3 .

$\therefore$ By theorem 2.3.1 , $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1+\frac{1}{n})^n$ exists .

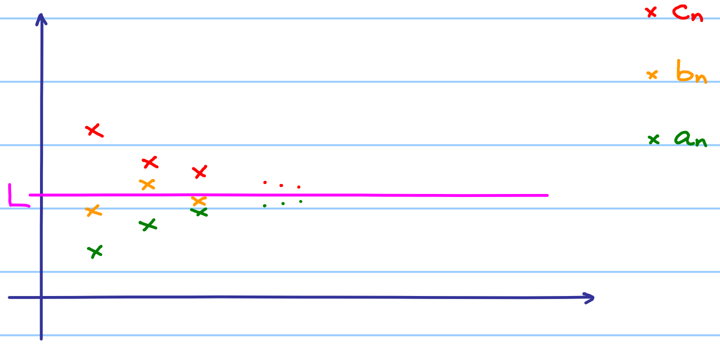
2.4 Sandwich Theorem

Theorem 2.4.1 (Sandwich Theorem)

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers.

If $a_n \le b_n \le c_n \ \forall n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Geometrical meaning:



In fact, the result is still true if $a_n \le b_n \le c_n$ for all $n \ge n_0$.

Idea: Estimate a sequence $\{b_n\}$ that we do not understand very well by sequences $\{a_n\}$ and $\{c_n\}$ that we understand well.

Example 2.4.1

Find $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

Note: $0 \leqslant \frac{1}{\sqrt{n+1}+\sqrt{n}} \leqslant \frac{1}{2\sqrt{n}} \quad \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$

By sandwich theorem, $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$

Example 2.4.2

Find $\lim_{n\to\infty} \frac{1}{n}\sin n$

Note: $-\frac{1}{n} \leqslant \frac{1}{n}\sin n \leqslant \frac{1}{n} \quad \forall n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$

By sandwich theorem, $\lim_{n\to\infty} \frac{1}{n}\sin n = 0$.

Exercise 2.4.1

Prove that $\lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0$

Hint: $-1 \leqslant (-1)^n \leqslant 1$

Exercise 2.4.2

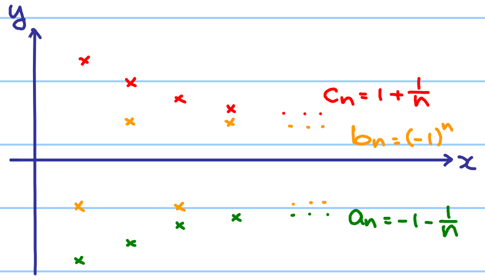
If $a_n \le b_n \le c_n$ for all $n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} a_n = -1$, $\lim_{n\to\infty} c_n = 1$,

can we conclude that $-1 \le \lim_{n\to\infty} b_n \le 1$?

No! Consider $a_n = -1 - \frac{1}{n}$, $b_n = (-1)^n$, $c_n = 1 + \frac{1}{n}$ for all $n \in \mathbb{Z}^+$.

We have $a_n \le b_n \le c_n$ for all $n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} a_n = -1$, $\lim_{n\to\infty} c_n = 1$,

however $\lim_{n\to\infty} b_n$ does not exist .



Theorem 2.4.2

Let $\{a_n\}$ be a sequence of real numbers .

$\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} |a_n| = 0$.

proof :

"$\Leftarrow$" Suppose that $\lim_{n\to\infty} |a_n| = 0$.

Note that $-|a_n| \le a_n \le |a_n| \ \forall n \in \mathbb{Z}^+$, and $\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = 0$,

by the sandwich theorem , $\lim_{n\to\infty} a_n = 0$

"$\Rightarrow$" Suppose that $\lim_{n\to\infty} a_n = 0$.

Then $\lim_{n\to\infty} a_n^2 = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} a_n) = 0 \cdot 0 = 0$

Note that $|a_n| = \sqrt{a_n^2}$

$$\therefore \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \sqrt{a_n^2} \overset{(*)}{=} \sqrt{\lim_{n\to\infty} a_n^2} = \sqrt{0} = 0$$

(*) is true because of $\sqrt{x}$ is a function that is continuous at 0

By using the above result, we obtain a result concerning a product of two sequences :

Theorem 2.4.3

If $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded , then $\lim_{n\to\infty} a_n b_n = 0$.

proof :

Note • $-|a_n| \leq a_n \leq |a_n| \quad \forall n \in \mathbb{Z}^+$

• $\{b_n\}$ is bounded $\Rightarrow \exists M>0$ s.t. $|b_n| \leq M$ (i.e. $-M \leq b_n \leq M$) $\forall n \in \mathbb{Z}^+$

$\therefore \ -M|a_n| \leq a_n b_n \leq M|a_n| \quad \forall n \in \mathbb{Z}^+$

$$\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} |a_n| = 0$$

$$\Rightarrow \lim_{n\to\infty} -M|a_n| = \lim_{n\to\infty} M|a_n| = 0$$

$\therefore$ By the sandwich theorem , $\lim_{n\to\infty} a_n b_n = 0$.

Theorem 2.4.4

Let $\alpha \in \mathbb{R}$, we have $\lim_{n\to\infty} \frac{\alpha^n}{n!} = 0$.

Remark : If $|\alpha| \leq 1$, the result is not surprising.

If $|\alpha| > 1$, α^n grows to infinity (if $\alpha < -1$, it is oscillating) as n goes to $+\infty$.

However , this theorem says that n! "grows faster" than α^n

proof :

Choose $K \in \mathbb{Z}^+$ such that $|\alpha| < K$ (i.e. $\frac{|\alpha|}{K} < 1$)

When $n > K$,

$$0 \leq \left|\frac{\alpha^n}{n!}\right| = \underbrace{\left(\frac{|\alpha|}{1} \frac{|\alpha|}{2} \frac{|\alpha|}{3} \cdot\cdot \frac{|\alpha|}{K-1}\right)}_{} \cdot \left(\frac{|\alpha|}{K} \frac{|\alpha|}{K-1} \cdots \frac{|\alpha|}{N}\right)$$

$$\leq M \left(\frac{|\alpha|}{K}\right)^{n-K+1}$$

$$\frac{|\alpha|}{K} < 1 \Rightarrow \lim_{n\to\infty} \left(\frac{|\alpha|}{K}\right)^{n-K+1} = 0 \Rightarrow \lim_{n\to\infty} M\left(\frac{|\alpha|}{K}\right)^{n-K+1} = 0$$

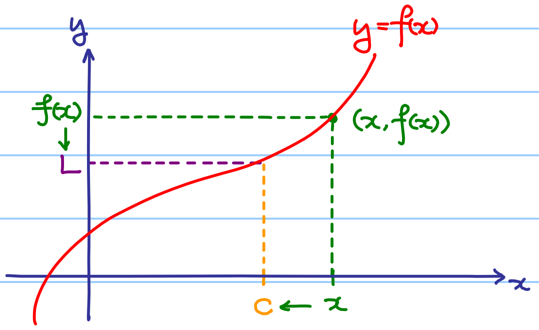
$\therefore$ By the sandwich theorem , $\lim_{n\to\infty} \left|\frac{\alpha^n}{n!}\right| = 0 \Leftrightarrow \lim_{n\to\infty} \frac{\alpha^n}{n!} = 0$

§3 Limits of Functions

3.1 Definition

Definition 3.1.1 (Informal)

If $f(x)$ gets closer and closer to a real number L as x gets closer and closer† to c from both sides, then L is called the limit of $f(x)$ at c, and we write $\lim_{x\to c} f(x) = L$.



† Note : a little bit misleading !

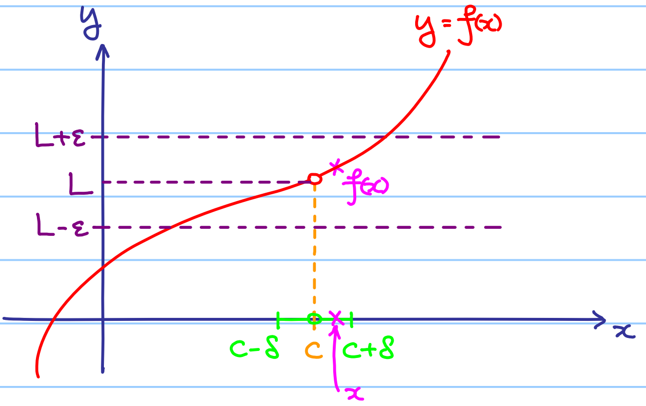
$f(c)$ may NOT equal to L, even it may be undefined !

Definition 3.1.2

Let $A \subseteq \mathbb{R}$, c be a cluster point of A and $f: A \to \mathbb{R}$ be a function

$L \in \mathbb{R}$ is said to be the limit of f at the point c if

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon \quad \forall x \in A$ with $0 < |x - c| < \delta$



Meaning : No matter how small ε is given,

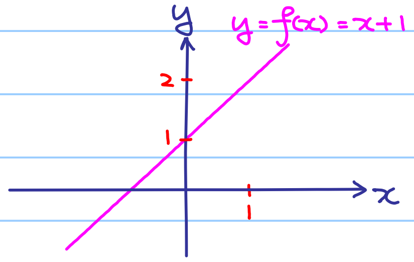
i.e. $x \neq c$

we can always find $\delta > 0$ s.t. if x is a point with $0 < \text{dist}(x, c) < \delta$

then $f(x)$ lies in the ε-tunnel (ε-neighborhood of L)

Example 3.1.1

If $f(x) = x+1$, find $\lim_{x\to 1} f(x)$.



+

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2	2 001	2.01	2.1

$f(x)$ tends to 2 as x tends to 1.

We write $\lim_{x\to 1} f(x) = 2$.

Remarks :

1) + The table only gives an intuitive idea, but NOT a rigorous proof !

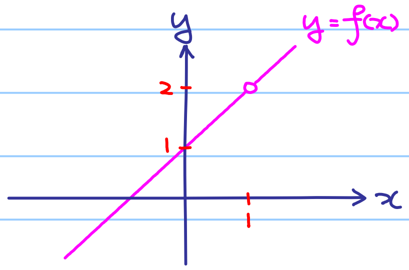
2) Always Remember :

Do NOT regard finding limit as putting $x=1$ into $f(x)$ and getting $f(1) = 2$!

Example 3.1.2

Let $f(x)$ be a function defined by $f(x) = \frac{x^2-1}{x-1}$, $x \neq 1$.

We can rewrite f as the following :



$$f(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	undefined	2 001	2.01	2.1

$f(x)$ tends to 2 as x tends to 1.

(But, we do NOT care what happens when $x=1$!)

We still have $\lim_{x\to 1} f(x) = 2$.

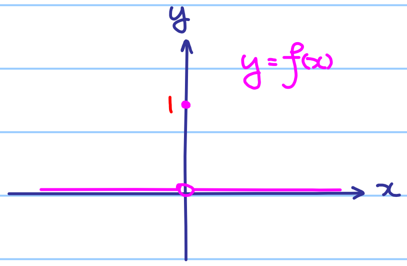
Compare with the previous example !

Idea : When x is "near" 1, both x^2-1 and $x-1$ are small,

but the quotient of them is not small !

Example 3.1.3

$$\text{Let } f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$



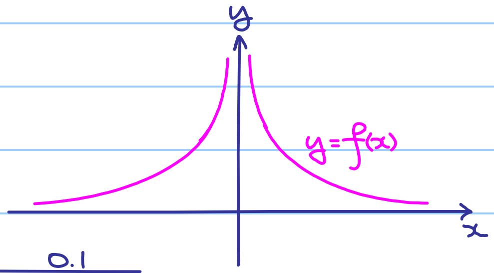
x	-0.1	-0.01	-0.001	$\boxed{0}$	0.001	0.01	0.1
$f(x)$	0	0	0	$\boxed{1}$	0	0	0

Do NOT care !

$\lim_{x \to 0} f(x) = 0$ which does NOT equal to $f(0) = 1$.

Example 3.1.4

Let $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x^2}$.



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	10^2	10^4	10^6	undefined	10^6	10^4	10^2

$f(x)$ tends to $+\infty$ (NOT a real number) as x tends to 0.

∴ $\lim_{x \to 0} f(x)$ does NOT exist.

(But some still write $\lim_{x \to 0} f(x) = +\infty$ or say $f(x)$ diverges to $+\infty$ as x tends to 0)

Theorem 3.1.1

1) If k is a constant, then $\lim_{x \to c} k = k$ regarded as constant function $f(x) = k$.

2) $\lim_{x \to c} x = c$.

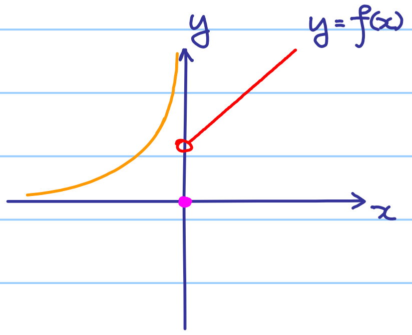
Definition 3.1.3 (Informal)

If $f(x)$ gets closer and closer to a real number L as x gets closer and closer to c from the right (left) hand side, then L is called the right (left) hand limit of $f(x)$ at c.

We denote it by $\lim_{x \to c^+} f(x) = L$ ($\lim_{x \to c^-} f(x) = L$).

Example 3.1.5

$$\text{If } f(x) = \begin{cases} x+1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ \frac{1}{x^2} & \text{if } x<0 \end{cases}$$



$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x+1 = 1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x^2}$ (does NOT exist)

$f(0) = 0$

Remark:

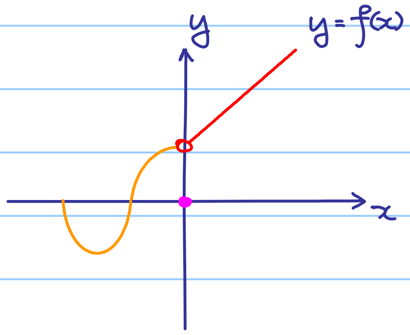
Right hand limit and left hand limit of a function at a point are NOT necessary to be same !

Theorem 3.1.2

$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$

Example 3.1.6

$$\text{If } f(x) = \begin{cases} x+1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ \cos x & \text{if } x<0 \end{cases}$$



$\underset{\substack{\uparrow \\ (x>0)}}{\lim_{x\to 0^+}} f(x) = \lim_{x\to 0^+} x+1 = 1$ $\qquad$ $\underset{\substack{\uparrow \\ (x<0)}}{\lim_{x\to 0^-}} f(x) = \lim_{x\to 0^-} \cos x = 1$

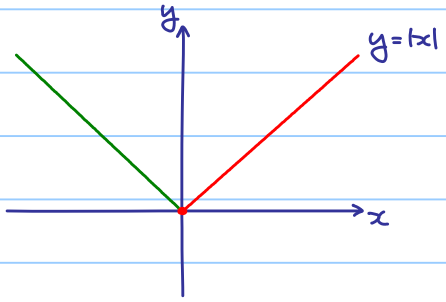
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 1 \Rightarrow \lim_{x\to 0} f(x)$ exists and $\lim_{x\to 0} f(x) = 1$

Remark: It is nothing related to $f(0) = 0$.

Example 3.1.7

Let $f(x) = |x|$

$$f(x) = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$



$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x = 0$

Don't skip!

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} -x = 0$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = 0$ and so $\lim\limits_{x \to 0} |x| = 0$.

Remark: We cannot say $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x$ or $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} -x$

since when we find $\lim\limits_{x \to 0} f(x)$, we need to consider the neighborhood of 0.

However, $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} x = 2$ since $f(x) = x$ in a neighborhood of 2.

3.2 Algebraic Properties of Limits

Theorem 3.2.1

If both $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to c} g(x)$ exist (Very important assumption!), then

(1) $\lim\limits_{x \to c} f(x) + g(x) = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$

(2) $\lim\limits_{x \to c} f(x) - g(x) = \lim\limits_{x \to c} f(x) - \lim\limits_{x \to c} g(x)$

(3) $\lim\limits_{x \to c} f(x) g(x) = \lim\limits_{x \to c} f(x) \lim\limits_{x \to c} g(x)$

(4) $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$ if $\lim\limits_{x \to c} g(x) \neq 0$

Example 3.2.1

Find $\lim\limits_{x \to 2} 3x^2 - 5$.

Logically:

① $\lim\limits_{x \to 2} x = 2$, so $\lim\limits_{x \to 2} x^2 = \lim\limits_{x \to 2} (x \cdot x) \overset{\text{By (3)}}{=} \lim\limits_{x \to 2} x \cdot \lim\limits_{x \to 2} x = 2 \cdot 2 = 4$

② $\lim\limits_{x \to 2} 3 = 3$, $\lim\limits_{x \to 2} x^2 = 4$, so $\lim\limits_{x \to 2} 3x^2 \overset{\text{By (3)}}{=} \lim\limits_{x \to 2} 3 \cdot \lim\limits_{x \to 2} x^2 = 3 \cdot 4 = 12$

③ $\lim\limits_{x \to 2} 3x^2 = 12$, $\lim\limits_{x \to 2} 5 = 5$, so $\lim\limits_{x \to 2} 3x^2 - 5 \overset{\text{By (2)}}{=} \lim\limits_{x \to 2} 3x^2 - \lim\limits_{x \to 2} 5 = 12 - 5 = 7$

But what we write:

$$\begin{aligned} \lim_{x \to 2} 3x^2 - 5 &= 3(\lim_{x \to 2} x)^2 - 5 \\ &= 3 \cdot 2^2 - 5 \\ &= 7 \end{aligned}$$

Example 3.2.2

Find $\lim_{x\to 1} \frac{3x^2-8}{x-2}$

$$\lim_{x\to 1} \frac{3x^2-8}{x-2} = \frac{\lim_{x\to 1} 3x^2-8}{\lim_{x\to 1} x-2} = \frac{3(\lim_{x\to 1} x)^2-8}{(\lim_{x\to 1} x)-2} = \frac{3\cdot 1^2-8}{1-2} = 5$$

Caution !

It seems that it makes no difference by putting $x=1$, and then

$$\lim_{x\to 1} \frac{3x^2-8}{x-2} = \frac{3\cdot 1^2-8}{1-2} = 5$$

But, think carefully ! Let $f(x) = \frac{3x^2-8}{x-2}$, how do you know $\lim_{x\to 1} f(x) = f(1)$?

Things will become clear when we discuss continuity of functions !

Example 3.2.3

Find $\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2}$

Note : $\lim_{x\to 1} x^2-3x+2 = 0$, so we cannot use (4).

$$\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x-2)} \underset{\uparrow}{=} \lim_{x\to 1} \frac{x+1}{x-2} \overset{(4)}{=} \frac{\lim_{x\to 1} x+1}{\lim_{x\to 1} x-2} = \frac{2}{-1} = -2$$

$\because x \neq 1$

$\therefore x-1 \neq 0$ and division can be done !

Example 3.2.4

Let $f : \mathbb{R}\setminus\{1\} \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{\sqrt{x}-1}{x-1}$. Find $\lim_{x\to 1} f(x)$.

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1}$$ (Something like rationalization)

$$= \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x}+1)}$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{x}+1}$$

$$= \frac{1}{2}$$

Example 3.2.5

$$\lim_{x\to 0} \frac{1}{x} = \lim_{x\to 0} x \cdot \frac{1}{x^2} \overset{(*)}{=} \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{1}{x^2} = 0 \cdot \lim_{x\to 0} \frac{1}{x^2} = 0$$ Anything wrong ?

$\lim_{x\to 0} \frac{1}{x^2}$ does NOT exist, so we cannot use (3) at (*).

Example 3.2.6

Suppose $f(0)=1$, $g(0)=2$, $\lim\limits_{x\to0}\frac{f(x)}{x}=3$ and $\lim\limits_{x\to0}\frac{g(x)}{x}=5$. Find $\lim\limits_{x\to0}\frac{f(x)}{g(x)}$ and $\lim\limits_{x\to0}f(x)$

$$\lim_{x\to0}\frac{f(x)}{g(x)}\overset{(*)}{=}\lim_{x\to0}\frac{\frac{f(x)}{x}}{\frac{g(x)}{x}}=\frac{\lim\limits_{x\to0}\frac{f(x)}{x}}{\lim\limits_{x\to0}\frac{g(x)}{x}}=\frac{3}{5}$$

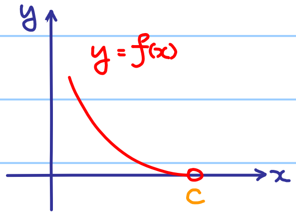
$$\lim_{x\to0}f(x)\overset{(*)}{=}\lim_{x\to0}\left(\frac{f(x)}{x}\cdot x\right)=\left(\lim_{x\to0}\frac{f(x)}{x}\right)\cdot\left(\lim_{x\to0}x\right)=3\cdot0=0$$

(*) Note $x\neq0$

Remark : Nothing related to $f(0)=1$, $g(0)=2$!

Theorem 3.2.2

If $f(x)\geq0$ for $x<c$ $(x>c)$ and $\lim\limits_{x\to c^-}f(x)$ $(\lim\limits_{x\to c^+}f(x))$ exists, then $\lim\limits_{x\to c^-}f(x)\geq0$ $(\lim\limits_{x\to c^+}f(x)\geq0)$.



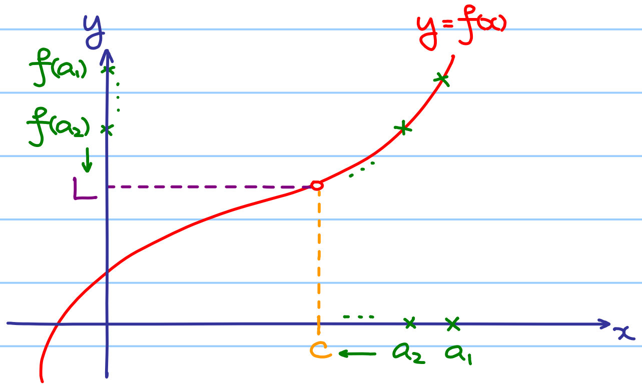
Combine them together.

if $f(x)\geq0$ for $x\neq c$ and $\lim\limits_{x\to c}f(x)$ exists, then $\lim\limits_{x\to c}f(x)\geq0$.

3.3 Relation Between Limits of Sequences and Functions

Theorem 3.3.1

$\lim\limits_{x\to c}f(x)=L \Leftrightarrow \forall$ sequence $\{a_n\}$ with $a_n\neq c$ $\forall n\in\mathbb{Z}^+$ and $\lim\limits_{n\to\infty}a_n=c$, we have $\lim\limits_{n\to\infty}f(a_n)=L$.



In fact , if we want to show $\lim\limits_{x\to c}f(x)=L$, it is quite impossible to check infinitely many sequences. This statement is useful in reverse direction :

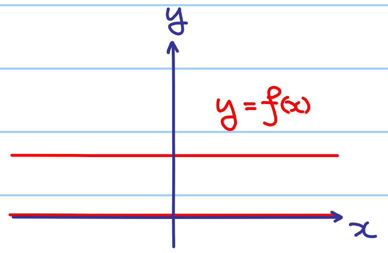
1) If $\exists$ $\{a_n\}$ with $a_n\neq c$ $\forall n\in\mathbb{Z}^+$ s.t $\lim\limits_{n\to\infty}a_n=c$, but $\lim\limits_{n\to\infty}f(a_n)$ does NOT exist,
 then $\lim\limits_{x\to c}f(x)$ does NOT exist .

2) If $\exists$ $\{a_n\}$, $\{b_n\}$ with $a_n, b_n\neq c$ $\forall n\in\mathbb{Z}^+$ s.t $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n=c$, but $\lim\limits_{n\to\infty}f(a_n)\neq\lim\limits_{n\to\infty}f(b_n)$
 then $\lim\limits_{x\to c}f(x)$ does NOT exist .

Example 3.3.1

Let $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} 1 & \text{if } x\in\mathbb{Q} \\ 0 & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$$



It seems the graph consists of two straight lines, but in fact infinitely many holes are there.

Consider sequences $\{a_n\}$, $\{b_n\}$ defined by

$a_n = \frac{1}{n}\in\mathbb{Q}$, $b_n=\frac{\sqrt{2}}{n}\in\mathbb{R}\setminus\mathbb{Q}$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, but $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} 1 = 1$

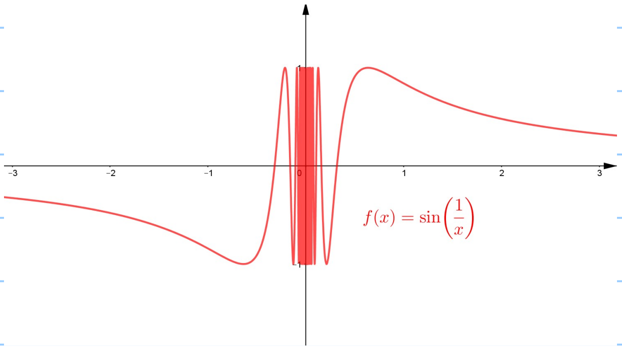
$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} 0 = 0$

$\therefore \lim_{x\to 0} f(x)$ does NOT exist.

Actually, with little modification, we can show $\lim_{x\to c} f(x)$ does NOT exist $\forall c\in\mathbb{R}$.

Example 3.3.2

Let $f(x)=\sin\frac{1}{x}$, for $x\neq 0$. Show that $\lim_{x\to 0} f(x)$ does NOT exist.



Solve $f(x)=1$

$\sin\frac{1}{x}=1$

$\frac{1}{x}=2n\pi+\frac{\pi}{2}$, $n\in\mathbb{Z}$

$x=\frac{1}{(2n+\frac{1}{2})\pi}$

$f(x)=-1$

$\sin\frac{1}{x}=-1$

$\frac{1}{x}=2n\pi+\frac{3\pi}{2}$, $n\in\mathbb{Z}$

$x=\frac{1}{(2n+\frac{3}{2})\pi}$

Consider sequences $\{a_n\}$, $\{b_n\}$ defined by

$a_n=\frac{1}{(2n+\frac{1}{2})\pi}$, $b_n=\frac{1}{(2n+\frac{3}{2})\pi}$ for $n\in\mathbb{Z}^+$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, but $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} 1 = 1$, $\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} -1 = -1$.

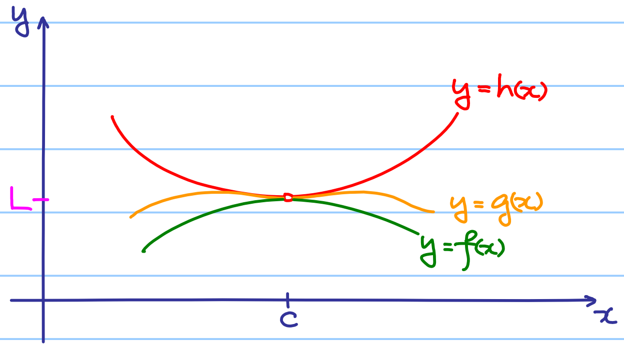
$\therefore \lim_{x\to 0} f(x)$ does NOT exist.

3.4 Sandwich Theorem for Functions

Theorem 3.4.1

If $f(x) \leq g(x) \leq h(x)$ $\forall x \in \mathbb{R} \setminus \{c\}$ and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$

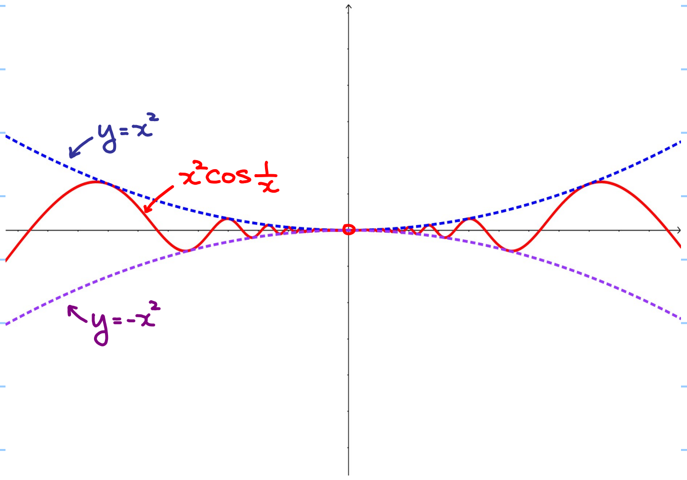
Geometrical meaning :



In fact, the result is still true if $f(x) \leq g(x) \leq h(x)$ holds in an open interval containing c but possibly except c

Example 3.4.1

Prove that $\lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$



Note that : $-1 \leq \cos \frac{1}{x} \leq 1$ for $x \neq 0$

$$-x^2 \leq x^2 \cos \frac{1}{x} \leq x^2$$

and $\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$

By sandwich theorem, $\lim_{x \to 0} x^2 \cos^2 \frac{1}{x} = 0$.

Theorem 3.4.2

$\lim_{x \to c} f(x) = 0 \iff \lim_{x \to c} |f(x)| = 0$

proof :

"$\Leftarrow$" Suppose that $\lim_{x \to c} |f(x)| = 0$

Note that $-|f(x)| \leq f(x) \leq |f(x)|$ $\forall x \in \mathbb{R} \setminus \{c\}$ and $\lim_{x \to c} -|f(x)| = \lim_{x \to c} |f(x)| = 0$

by the sandwich theorem, $\lim_{x \to c} f(x) = 0$

"$\Rightarrow$" Suppose that $\lim_{x \to c} f(x) = 0$.

Then $\lim_{x \to c} [f(x)]^2 = (\lim_{x \to c} f(x)) \cdot (\lim_{x \to c} f(x)) = 0 \cdot 0 = 0$

Note that $|f(x)| = \sqrt{[f(x)]^2}$

$$\therefore \lim_{x \to c} |f(x)| = \lim_{x \to c} \sqrt{[f(x)]^2}$$

$$\overset{(*)}{=} \sqrt{\lim_{x \to c} [f(x)]^2}$$

$$= \sqrt{0}$$

$$= 0$$

(*) is true because of $\sqrt{x}$ is a function that is continuous at 0

Example 3.4.2

By considering the previous theorem with $f(x) = x$, we have $\lim_{x \to 0} x = 0 \Leftrightarrow \lim_{x \to 0} |x| = 0$

Compare with example 3.1.7.

Example 3.4.3

Prove that $\lim_{x \to 0} x \cos \frac{1}{x} = 0$

Note that $-1 \leq \cos \frac{1}{x} \leq 1 \quad \forall x \in \mathbb{R} \setminus \{0\}$

$-|x| \leq x \leq |x| \quad \forall x \in \mathbb{R}$

$\therefore \quad -|x| \leq x \cos \frac{1}{x} \leq |x| \quad \forall x \in \mathbb{R} \setminus \{0\}$

Also $\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0$,

by the sandwich theorem, $\lim_{x \to 0} x \cos \frac{1}{x} = 0$

Remark:

Sandwich theorem can be generalized to left and right hand limit.

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions and $c \in \mathbb{R}$

If $f(x) \leq g(x) \leq h(x)$ for all $x < c$ $(x > c)$ and $\lim_{x \to c^-} f(x) = \lim_{x \to c^-} h(x) = L$ $\left(\lim_{x \to c^+} f(x) = \lim_{x \to c^+} h(x) = L \right)$

then $\lim_{x \to c^-} g(x) = L$ $\left(\lim_{x \to c^+} g(x) = L \right)$.

Exercise 3.4.1

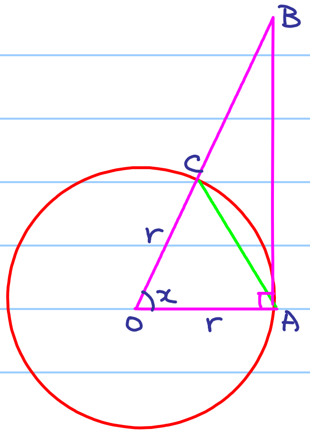
Prove that $\lim_{x \to 1^+} (x^2 - 1) \sin\left(\frac{1}{\sqrt{x-1}} \right) = 0$

Theorem 3.4.3

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Idea: When x becomes small (but not zero), both $\sin x$ and x are small, but the quotient of them is not small!

proof.



1) Consider $0<x<\frac{\pi}{2}$, we have

Area of $\triangle OAC$ < Area of sector OAC < Area of $\triangle OAB$

$$\frac{1}{2}r^2\sin x<\frac{1}{2}r^2x<\frac{1}{2}r^2\tan x$$

$$\sin x<x<\tan x$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{\sin x}{x}<1 \qquad\qquad \cos x<\frac{\sin x}{x}$$

$$\therefore\ \cos x<\frac{\sin x}{x}<1 \quad \text{for } 0<x<\frac{\pi}{2},$$

Also, $\lim_{x\to 0^+}\cos x=\lim_{x\to 0^+}1=1$, therefore by the sandwich theorem, $\lim_{x\to 0^+}\frac{\sin x}{x}=1$.

2) Consider $-\frac{\pi}{2}<x<0$, we have

Let $y=-x$, then $0<y<\frac{\pi}{2}$, so

$$\cos y<\frac{\sin y}{y}<1$$

$$\cos(-x)<\frac{\sin(-x)}{-x}<1$$

$$\therefore\ \cos x<\frac{\sin x}{x}<1 \quad \text{for } -\frac{\pi}{2}<x<0.$$

Also, $\lim_{x\to 0^-}\cos x=\lim_{x\to 0^-}1=1$, therefore by the sandwich theorem, $\lim_{x\to 0^-}\frac{\sin x}{x}=1$.

$\therefore$ By (1) and (2), $\lim_{x\to 0^+}\frac{\sin x}{x}=\lim_{x\to 0^-}\frac{\sin x}{x}=1$, therefore $\lim_{x\to 0}\frac{\sin x}{x}=1$.

Example 3.4.4

Find $\lim_{x\to 0}\frac{\sin 3x}{2x}$.

$$\lim_{x\to 0}\frac{\sin 3x}{2x}=\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{3}{2}=1\cdot\frac{3}{2}=\frac{3}{2}$$

Example 3.4.5

Find $\lim_{x\to 0}\frac{\cos ax-\cos bx}{x^2}$

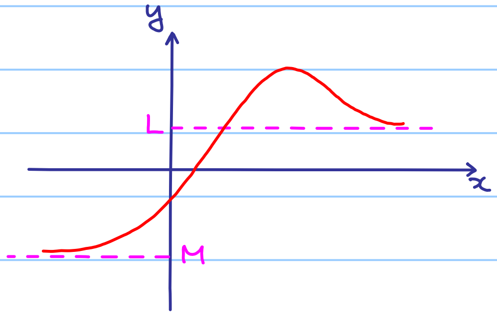
$$\begin{aligned}\lim_{x\to 0}\frac{\cos ax-\cos bx}{x^2}&=\lim_{x\to 0}\frac{2\sin\frac{a+b}{2}x\,\sin\frac{b-a}{2}x}{x^2}\\&=\lim_{x\to 0}2\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)\frac{\sin\frac{a+b}{2}x}{\frac{a+b}{2}x}\,\frac{\sin\frac{b-a}{2}x}{\frac{b-a}{2}x}\\&=\frac{b^2-a^2}{2}\end{aligned}$$

3.5 Limits at Infinity

Definition 3.5.1 (Informal)

If $f(x)$ gets closer and closer to a real number L as x gets bigger and bigger (as x goes to $+\infty$), then L is called the limit of $f(x)$ at $+\infty$ We write $\lim_{x\to+\infty} f(x) = L$.

(Similar definition for $\lim_{x\to-\infty} f(x)$)



From the graph, we have

$\lim_{x\to+\infty} f(x) = L$ but $\lim_{x\to-\infty} f(x) = M$.

$\therefore \lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are NOT necessary to be the same !

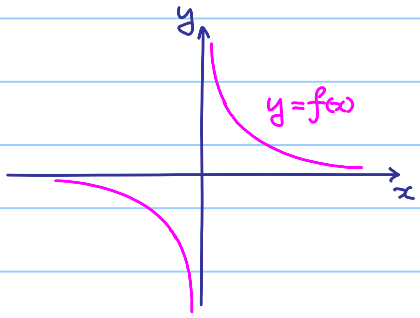
However if $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = L$, some simply write $\lim_{x\to\pm\infty} f(x) = L$.

Example 3.5.1

Let $f(x) = \frac{1}{x}$, $x \neq 0$.

Then $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$,

or simply write $\lim_{x\to\pm\infty} f(x) = 0$



Theorem 3.5.1

1) If $k>0$, then $\lim_{x\to+\infty} \frac{1}{x^k} = 0$

2) $\lim_{x\to+\infty} (1+\frac{1}{x})^x = e$

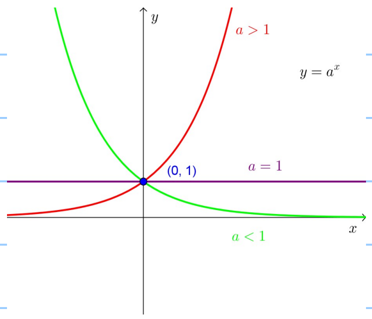
(NOT surprising as $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$)

Theorem 3.5.2

If $a>1$, $\lim_{x\to-\infty} a^x = 0$

If $1>a>0$, $\lim_{x\to+\infty} a^x = 0$

$\lim_{x\to\pm\infty} 1^x = 1$



3.6 Algebraic Properties of Limits at Infinity

Theorem 3.6.1

If both $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} g(x)$ exist (Very important assumption!), then

(1) $\lim_{x\to+\infty} f(x)+g(x) = \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty} g(x)$

(2) $\lim_{x\to+\infty} f(x)-g(x) = \lim_{x\to+\infty} f(x) - \lim_{x\to+\infty} g(x)$

(3) $\lim_{x\to+\infty} f(x)g(x) = \lim_{x\to+\infty} f(x) \lim_{x\to+\infty} g(x)$

(4) $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to+\infty} f(x)}{\lim_{x\to+\infty} g(x)}$ if $\lim_{x\to+\infty} g(x) \neq 0$

Similar results hold for limits at $-\infty$.

Example 3.6.1

Find $\lim_{x\to+\infty} \frac{3x^2}{x^2+x+1}$

$$\lim_{x\to+\infty} \frac{3x^2}{x^2+x+1} \neq \frac{\lim_{x\to+\infty} 3x^2}{\lim_{x\to+\infty} x^2+x+1}$$

Both limits does NOT exist.

$$= \lim_{x\to+\infty} \frac{3}{1+\frac{1}{x}+\frac{1}{x^2}}$$

$$= \lim_{x\to+\infty} \frac{3}{1+0+0}$$

$$= 3$$

Example 3.6.2

Find $\lim_{x\to+\infty} \frac{2x+1}{3x^2-2x+1}$

$$\lim_{x\to+\infty} \frac{2x+1}{3x^2-2x+1}$$

$$= \lim_{x\to+\infty} \frac{\frac{2}{x}+\frac{1}{x^2}}{3-\frac{2}{x}+\frac{1}{x^2}}$$

$$= \frac{0+0}{3+0+0}$$

$$= 0$$

Conclusion :

If $p(x)$ and $q(x)$ are polynomials

$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$ with $a_m \neq 0$ (i.e. $\deg p(x) = m$)

$q(x) = b_n x^n + a_{n-1} x^{n-1} + \cdots + b_1 x + b_0$ with $b_n \neq 0$ (i.e. $\deg q(x) = n$)

then

$$\lim_{x\to+\infty} \frac{p(x)}{q(x)} = \begin{cases} +\infty/-\infty & \text{if } m > k \\ \frac{a_m}{b_m} & \text{if } m = k \\ 0 & \text{if } m < k \end{cases}$$

Similar result as the case in limits of sequences !

Example 3.6.3

Find $\lim\limits_{x\to-\infty} \frac{x}{\sqrt{4x^2+1}}$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{4x^2+1}}$$

deg 1

roughly, deg 1

$\Rightarrow$ limit should exist !

$$= \lim_{x\to-\infty} \frac{1}{\frac{1}{x}\sqrt{4x^2+1}}$$

$$= \lim_{x\to-\infty} \frac{1}{-\sqrt{\frac{1}{x^2}}\cdot\sqrt{4x^2+1}}$$

(Caution : $x < 0 \Rightarrow \frac{1}{x} = -\sqrt{\left(\frac{1}{x}\right)^2} = -\sqrt{\frac{1}{x^2}}$)

$$= \lim_{x\to-\infty} -\frac{1}{\sqrt{4+\frac{1}{x^2}}}$$

$$= -\frac{1}{2}$$

Following this idea, we are going to compare exponential functions and polynomials.

Theorem 3.6.2

1) $\lim\limits_{x\to+\infty} x^k e^{-x} = \lim\limits_{x\to+\infty} \frac{x^k}{e^x} = 0$, for any $k > 0$.

2) $\lim\limits_{x\to+\infty} p(x) e^{-x} = \lim\limits_{x\to+\infty} \frac{p(x)}{e^x} = 0$, for any polynomial $p(x)$.

Roughly speaking : As $x \to +\infty$, e^x grows "faster" than any polynomial

Proof can be done when L'Hôptial's rule is covered.

Example 3.6.4

Find $\lim_{x\to+\infty}\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$ and $\lim_{x\to-\infty}\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

$$\lim_{x\to+\infty}\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$$ dominating terms

$$=\lim_{x\to+\infty}\frac{1+e^{-2x}}{1-e^{-2x}}$$

$$=\frac{1+0}{1-0}$$

$$=1$$

$$\lim_{x\to-\infty}\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$$ dominating terms

$$=\lim_{x\to+\infty}\frac{e^{2x}+1}{e^{2x}-1}$$

$$=\frac{0+1}{0-1}$$

$$=-1$$

Idea: Taking quotient of the dominating term.

3.7 Limits Involving e

Example 3.7.1

Find $\lim_{x\to+\infty}\left(1+\frac{1}{2x-1}\right)^{x}$

$$\lim_{x\to+\infty}\left(1+\frac{1}{2x-1}\right)^{x}=\lim_{x\to+\infty}\left(1+\frac{1}{2x-1}\right)^{\frac{1}{2}(2x-1)+\frac{1}{2}}$$

$$=\lim_{x\to+\infty}\left[\left(1+\frac{1}{2x-1}\right)^{2x-1}\right]^{\frac{1}{2}}\cdot\left(1+\frac{1}{2x-1}\right)^{\frac{1}{2}}$$

$$=e^{\frac{1}{2}}\cdot 1$$

$$=e^{\frac{1}{2}}$$

Example 3.7.2

Find $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}$

Let $y=-x$, as $x\to-\infty$, $y\to+\infty$

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{y\to+\infty}\left(1-\frac{1}{y}\right)^{-y}$$

$$=\lim_{y\to+\infty}\left(\frac{y}{y-1}\right)^{y}$$

$$=\lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\cdot\left(1+\frac{1}{y-1}\right)$$

$$=e\cdot 1$$

$$=e$$

Remark: From the above example, we know $\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^{x}=e$.

Example 3.7.3

Find $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}}$.

Let $y = \frac{1}{x}$, as $x \to 0$, $y \to \pm\infty$ (Not only $+\infty$, but also $-\infty$)

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = \lim_{y\to\pm\infty} (1+\frac{1}{y})^{y} = e$$

Next, consider $\lim\limits_{x\to 0} \frac{e^x-1}{x}$

Idea: When x becomes small (but not zero), both e^x-1 and x are small,

but the quotient of them is not small!

Theorem 3.7.1

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1.$$

Cheating: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} \frac{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}{x} = \lim_{x\to 0} 1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots \overset{(*)}{=} 1$$

(*) is cheating since we are summing up infinitely many small terms, so algebraic properties of limits (theorem 3.2.1) cannot be applied.

Exercise 3.7.1

Use a calculator and fil the following table to convince yourself that $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{e^x-1}{x}$				undefined			

Example 3.7.2

Find $\lim\limits_{x\to 0} \frac{e^{3x}-1}{2x}$.

$$\lim_{x\to 0} \frac{e^{3x}-1}{2x} = \lim_{x\to 0} \frac{e^{3x}-1}{3x} \cdot \frac{3}{2}$$

$$= \left(\lim_{x\to 0} \frac{e^{3x}-1}{3x}\right) \cdot \left(\lim_{x\to 0} \frac{3}{2}\right)$$

$$= 1 \cdot \frac{3}{2}$$

$$= \frac{3}{2}$$

3.8 Sandwich Theorem at Infinity

Theorem 3.8.1

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions.

If $f(x) \le g(x) \le h(x)$ for all $x \in \mathbb{R}$ (actually: $[a, +\infty)$ is sufficient)

and $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} h(x) = L$, then $\lim_{x \to +\infty} g(x) = L$.

Geometrical meaning:

Similar result holds for limits at $-\infty$

Example 3.8.1

Find $\lim_{x \to +\infty} e^{-x} \sin x$

Since $-1 \le \sin x \le 1$ and $e^{-x} > 0$

$$-e^{-x} \le e^{-x} \sin x \le e^{-x}$$

Note: $\lim_{x \to +\infty} -e^{-x} = \lim_{x \to +\infty} e^{-x} = 0$.

By the sandwich theorem, $\lim_{x \to +\infty} e^{-x} \sin x = 0$.

Exercise 3.8.1

Show that $\lim_{x \to +\infty} \frac{\sin x}{x} = 0$.

(Don't mix up with $\lim_{x \to 0} \frac{\sin x}{x} = 1$)

§4 Continuity

4.1 Definition

Definition 4.1.1

Let $c \in A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$ be a function.

A function $f(x)$ is said to be continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$.

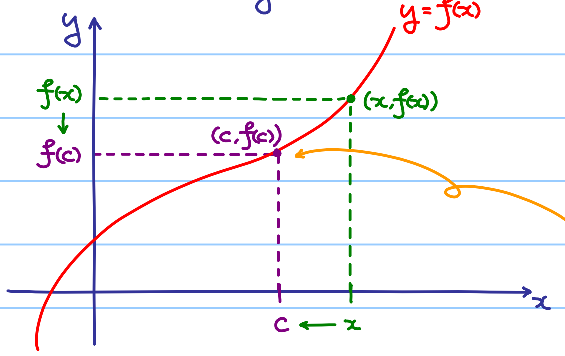
Idea:

③ they equal

$$\lim_{x \to c} f(x) = f(c)$$

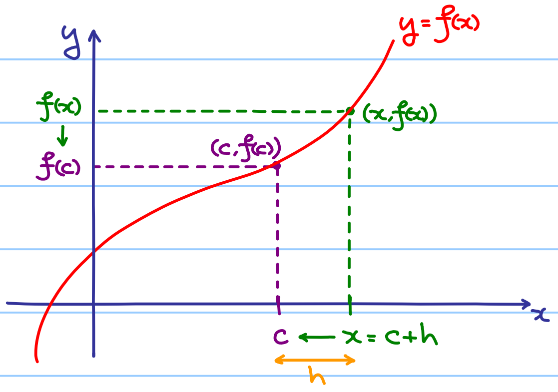
① limit exists ② well-defined

Geometrical meaning:



the curve does NOT break up at the point $x = c$!

Furthermore, if a function $f: A \to \mathbb{R}$ is continuous at every point in A, then f is said to be continuous on A.



Let $h = x - c$, i.e. $x = c + h$ (Remark: When $x < c$, we have $h < 0$.)

When x tends to c, h tends to 0.

Therefore, we have another formulation:

A function $f(x)$ is said to be continuous at $x = c$ if $\lim_{h \to 0} f(c+h) = f(c)$.

4.2 Examples

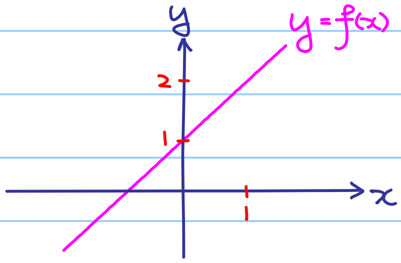
Example 4.2.1

Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x)=x+1$.

We have : ① $\lim_{x\to 1} f(x) = \lim_{x\to 1} x+1 = 2$

② $f(1)=(1)+1=2$

$\therefore$ f is continuous at $x=1$.



Example 4.2.2

Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by

$$f(x)=\begin{cases}\frac{\sin x}{x} & \text{if } x\neq 0\\ a & \text{if } x=0\end{cases}$$

We have $\underset{\substack{\uparrow\\ \text{i.e. } x\neq 0}}{}$ $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin x}{x}=1$

$\therefore$ f is continuous at $x=0 \Leftrightarrow \lim_{x\to 0} f(x)=f(0)$, i.e. $a=1$

Recall :

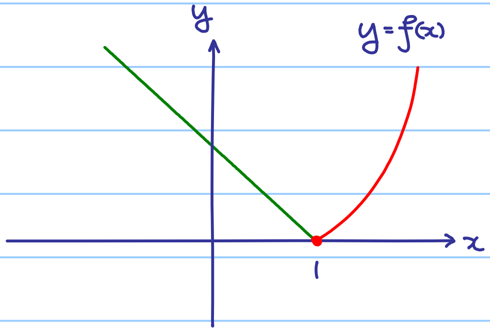
$\lim_{x\to c} f(x)=L$ if and only if $\lim_{x\to c^-} f(x)=\lim_{x\to c^+} f(x)=L$

Rewrite :

A function $f(x)$ is said to be continuous at $x=c$ if $\lim_{x\to c^-} f(x)=\lim_{x\to c^+} f(x)=f(c)$

Example 4.2.3

If $f(x)=\begin{cases}x^2-1 & \text{if } x\geq 1\\ 1-x & \text{if } x<1\end{cases}$



① $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2-1 = 0$

② $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1-x = 0$

③ $f(1)=1^2-1=0$

$\therefore$ f is continuous at $x=1$.

Example 4.2.4

Absolute Value : $|x| = \sqrt{x^2}$

For example:

$|3| = \sqrt{3^2} = \sqrt{9} = 3$

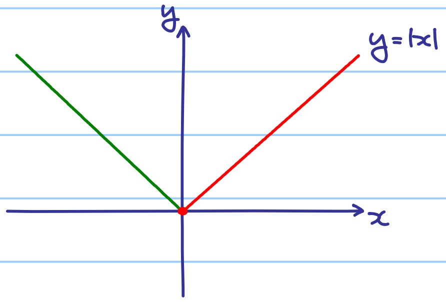
$|0| = \sqrt{0^2} = \sqrt{0} = 0$

$|-3| = \sqrt{(-3)^2} = \sqrt{9} = 3$

(Simply speaking: throw away the negative sign)

Rewrite $|x|$ as a piecewise defined function:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



We have :

① $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$

② $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x = 0$

③ $f(0) = 0$

$\therefore$ $|x|$ is continuous at $x=0$

Exercise 4.2.1

Show that $f(x)=|x|$ is a continuous function.

Hint: Show that $f(x)$ is continuous

(i) for $x>0$; (ii) at $x=0$; (iii) for $x<0$.

Theorem 4.2.1

- If $f(x)$ and $g(x)$ are continuous at $x=c$, then $f(x) \pm g(x)$, $f(x)g(x)$, $\frac{f(x)}{g(x)}$ $(g(c) \neq 0)$ are continuous at $x=c$ as well.
- Polynomial functions and exponential functions are continuous everywhere.
- Trigonometric functions and logarithmic functions are continuous at every point where they are defined.
- If $g(x)$ is continuous at $x=c$ and $f(x)$ is continuous at $x=g(c)$, then $f(g(x))$ is continuous at $x=c$.

(That's why we usually have $\lim_{x\to c} f(x) = f(c)$ as we usually looking at continuous functions.)

Example 4.2.5

Let $f(x) = \frac{2x^2+3}{x^2-3x+2}$ quotient of two polynomials (continuous functions)

$= \frac{2x^2+3}{(x-2)(x-1)}$ the denominator is nonzero when $x \neq 1$ or 2.

$\therefore$ $f(x)$ is continuous at $x \in \mathbb{R} \setminus \{1,2\}$

Example 4.2.6

Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that

(i) f is continuous at 0.

(ii) $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Show that :

a) $f(0) = 0$;

b) f is continuous everywhere.

proof :

a) Putting $x = y = 0$,

$$f(0+0) = f(0) + f(0)$$
$$f(0) = 2f(0)$$
$$f(0) = 0$$

b) f is continuous at 0 $\Rightarrow \lim\limits_{h \to 0} f(0+h) = f(0)$

$\Rightarrow \lim\limits_{h \to 0} f(h) = f(0) = 0$

Let $x_0 \in \mathbb{R}$.

$$\lim_{h \to 0} f(x_0+h) = \lim_{h \to 0} [f(x_0) + f(h)]$$ (Property of f)

$$= f(x_0) + \lim_{h \to 0} f(h)$$
$$= f(x_0)$$

$\therefore$ f is continuous everywhere.

4.3 Sequential Criterion for Continunity

Theorem 4.3.1

A function f is continuous at $x=c$ if and only if for every sequence $\{a_n\}$ with $a_n \neq c \ \forall n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} a_n = c$, we have $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(c)$.

Example 4.3.1

Think : Find $\lim_{n\to\infty} \sqrt{\frac{n^2+1}{4n^2+3}}$

How did we do ?

① $\lim_{n\to\infty} \frac{n^2+1}{4n^2+3} = \frac{1}{4}$

② $\lim_{n\to\infty} \sqrt{\frac{n^2+1}{4n^2+3}} \overset{(*)}{=} \sqrt{\lim_{n\to\infty} \frac{n^2+1}{4n^2+3}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

so in general, (*) means $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$, why it is true ?

Consider $a_n = \frac{n^2+1}{4n^2+3}$, we have $\lim_{n\to\infty} a_n = \frac{1}{4}$

Also, we know $f(x) = \sqrt{x}$ is continuous at $\frac{1}{4}$,

$\therefore \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

$\lim_{n\to\infty} \sqrt{\frac{n^2+1}{4n^2+3}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

Example 4.3.2

Consider

$$f(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{if } x\neq 0 \end{cases}, \quad \text{and } a_n = \frac{1}{n}.$$

Note : f is NOT continuous at $x=0$.

$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(\frac{1}{n}) = \lim_{n\to\infty} 0 = 0$

$f(\lim_{n\to\infty} a_n) = f(0) = 1$

$\therefore \lim_{n\to\infty} f(a_n) \neq f(\lim_{n\to\infty} a_n)$

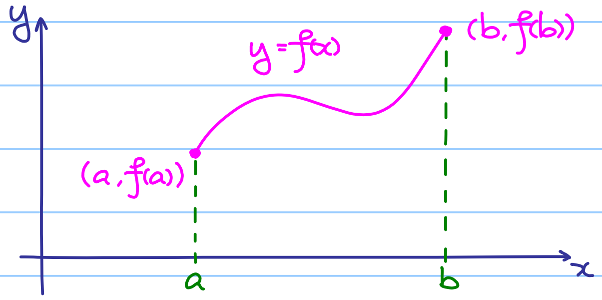
4.4 Continuity on $[a,b]$

Definition 4.4.1

Let $f : [a,b] \longrightarrow \mathbb{R}$

f is said to be continuous at $x=a$ if $\lim_{x \to a^+} f(x) = f(a)$;

f is said to be continuous at $x=b$ if $\lim_{x \to b^-} f(x) = f(b)$.



(We cannot talk about $\lim_{x \to a^-} f(x)$ and $\lim_{x \to b^+} f(x)$!)

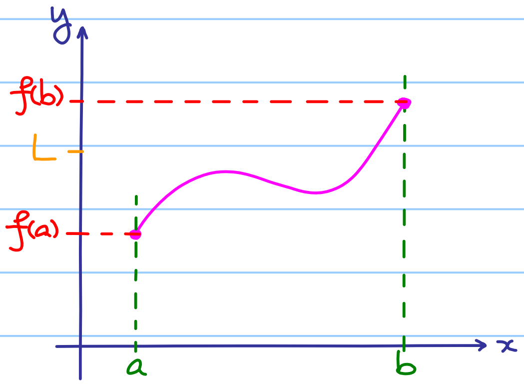
Furthermore, if a function $f : [a,b] \to \mathbb{R}$ is continuous at every point $x \in [a,b]$, then f is said to be continuous on $[a,b]$.

Theorem 4.4.1 (Intermediate Value Theorem)

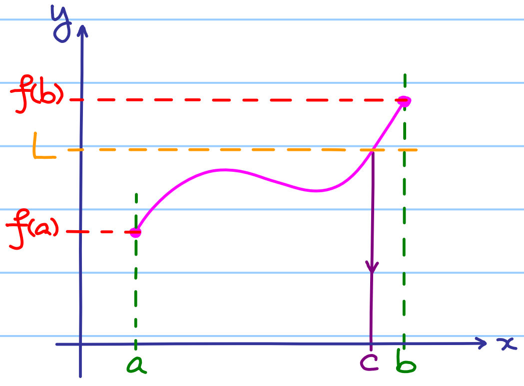
Suppose that f is continuous on $[a,b]$ and $f(a) < f(b)$.

Furthermore, if $L \in \mathbb{R}$ such that $f(a) < L < f(b)$.

then there exists (at least one) $c \in (a,b)$ such that $f(c) = L$.



$\Rightarrow$



Similar result holds for $f(a) > L > f(b)$. (What is the picture ?)

Example 4.4.1

Let $f(x) = x^2$

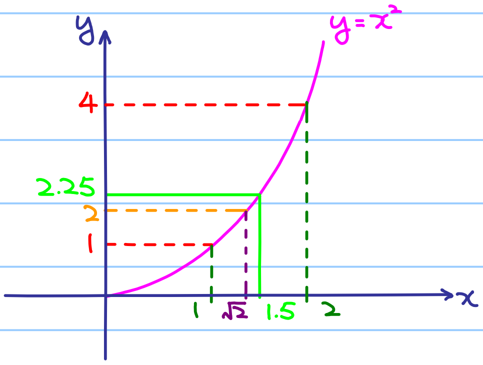
① $f(1) = 1 < 2 < 4 = f(2)$

② f is continuous on $[1,2]$ (In fact, on $\mathbb{R}$)

By the Intermediate Value Theorem, there exists $c \in (1,2)$ such that $f(c) = c^2 = 2$.

c is $\sqrt{2}$ by definition!

$\therefore\ 1 < \sqrt{2} < 2$ (estimates $\sqrt{2}$)



We can further obtain a better estimation by:

① Take the mid-point of $[1,2]$, i.e. 1.5 .

② $f(1.5) = 2.25 > 2$.

③ $f(1) = 1 < 2 < 2.25 = f(1.5)$

. $1 < \sqrt{2} < 1.5$

Repeating again and again to obtain better and better estimation.

It is well-known as method of bisection!

Example 4.4.2

Show that $2^x = \frac{1}{x^2}$ has at least one solution

(i.e. let $f(x) = 2^x - \frac{1}{x^2}$, the equation $f(x) = 0$ has at least one solution)

Note that $f(\frac{1}{2}) = 2^{\frac{1}{2}} - \frac{1}{(\frac{1}{2})^2} = \sqrt{2} - 4 < 0$

$f(1) = 2^1 - \frac{1}{1^2} = 2 - 1 = 1 > 0$

and f is continuous on $[\frac{1}{2}, 1]$.

By the Intermediate Value Theorem, there exists $c \in (\frac{1}{2}, 1)$

such that $f(c) = 2^c - \frac{1}{c^2} = 0$, i.e. $2^c = \frac{1}{c^2}$.

Remark: $\frac{1}{2}$ and 1 can be replaced by other points a and b, but we have to make sure that f is continuous on $[a,b]$.

$f(-1) = 2^{-1} - \frac{1}{(-1)^2} = \frac{1}{2} - 1 = -\frac{1}{2} < 0$

$f(1) = 2^1 - \frac{1}{1^2} = 2 - 1 = 1 > 0$

Can we use the Intermediate Value Theorem? No! f is NOT continuous on $[-1,1]$!

Example 4.4.3

Let $f(x) = x^3 + bx^2 + cx + d$ where $b, c, d \in \mathbb{R}$.

Prove that the equation $f(x) = 0$ has at least one real root.

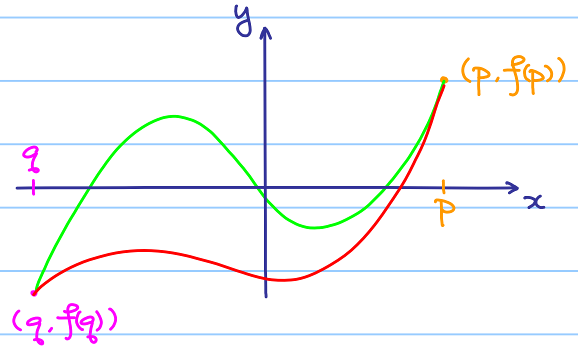
$f(x) = x^3 + bx^2 + cx + d$

$= x^3 \left(1 + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3}\right)$

We can choose $p > 0$ such that if $x = p$, $1 + \frac{b}{p} + \frac{c}{p^2} + \frac{d}{p^3} > 0$

Similarly, we can choose $q < 0$ such that if $x = q$, $1 + \frac{b}{q} + \frac{c}{q^2} + \frac{d}{q^3} > 0$

Then $f(q) < 0 < f(p)$.



What is the graph of $y = f(x)$?

Red? Green?

Anyway, they cut the x-axis!

f is continuous on $[q, p]$.

$\therefore$ By Intermediate Value Theorem, there exists $x_0 \in (q, p)$ such that $f(x_0) = 0$

Remark:

1) By factor theorem, $(x - x_0)$ is a factor of $f(x)$.

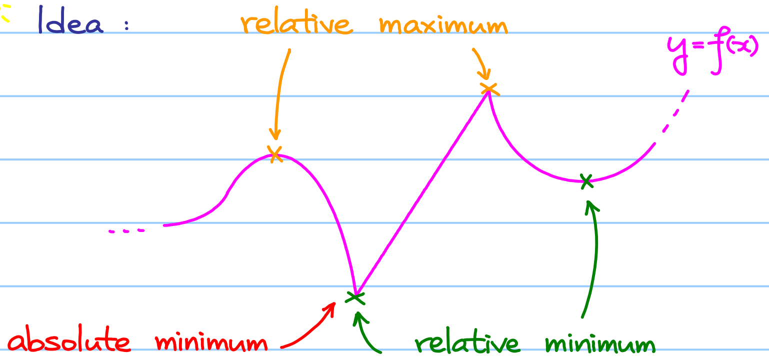
2) This idea can be generalized to any polynomial $f(x)$ with odd degree.

4.5 Relative and Absolute Extrema

Definition 4.5.1

f has an absolute maximum (resp. minimum) point at a if
$f(x) \le f(a)$ (resp. $f(x) \ge f(a)$) for all x in the domain of f.
f has a relative maximum (resp. minimum) point at a if
$f(x) \le f(a)$ (resp. $f(x) \ge f(a)$) for all x in a neighborhood of a.

Idea :



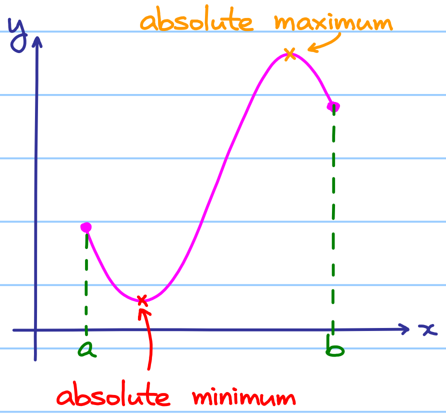
Note : No absolute maximum in this case.

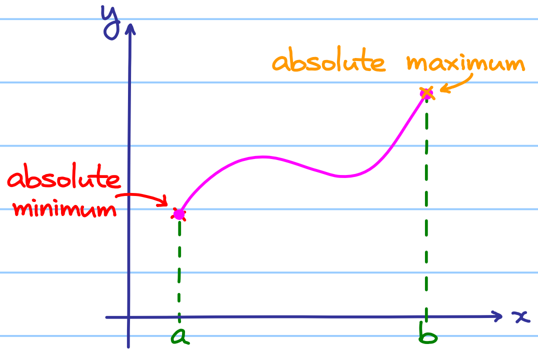
Remark :

1) We simply use maximum / minimum to refer relative maximum / minimum.
2) Absolute maximum / minimum are also called global maximum / minimum.

Theorem 4.5.1 (Maximum-Minimum Theorem)

Let $f:[a,b]\to\mathbb{R}$ be a continuous function
Then there exists $x_m, x_M \in [a,b]$ such that $f(x_m) \le f(x) \le f(x_M)$ for all $x\in[a,b]$.





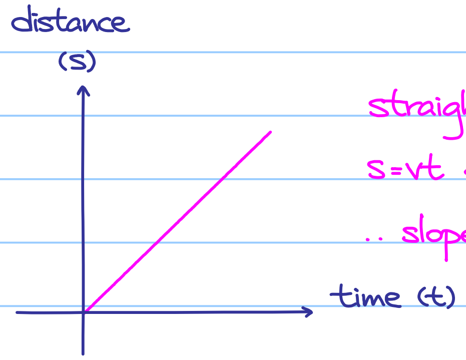
Absolute maximum / minimum may be attained at the boundary points of $[a,b]$.

Main question : Given a function, how to find all absolute / relative extrema ?
Differentiation provides a powerful tool for that.

§ 5 Differentiation

5.1 Idea of Derivative

Recall : (average) speed = $\frac{\text{distance}}{\text{time}}$



straight line

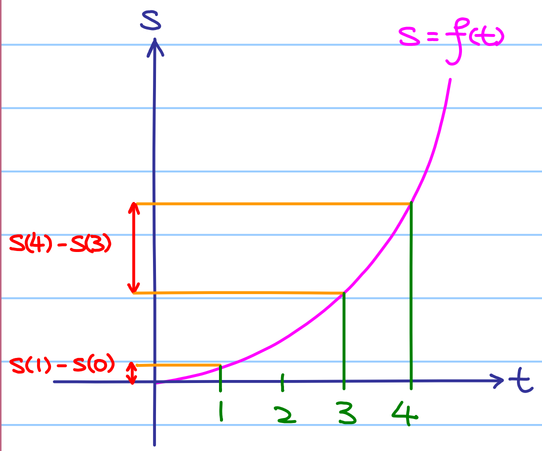
$s = vt \Leftrightarrow v = \frac{s}{t}$

$\therefore$ slope = v = speed .

Remark :

Using displacement and velocity if you know .

How about this case ?



distance traveled from t=0 to t=1 (S(1) - S(0))	<	distance traveled from t=3 to t=4 (S(4) - S(3))

Why ? The speed is changing .

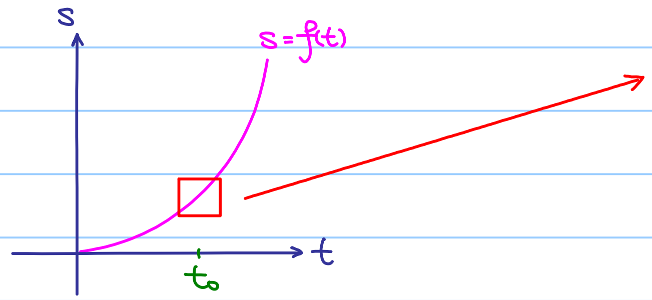
Speed is different at different moment .

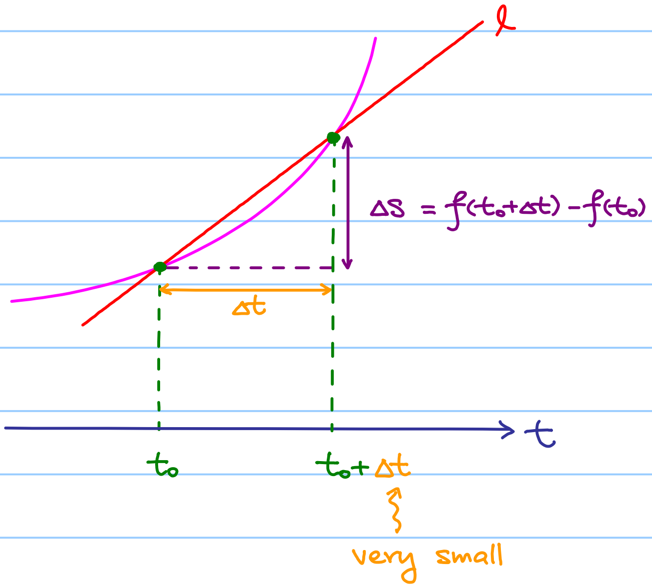
Hold on !

What is the meaning of speed at a particular moment (instantaneous speed) ?

We need a definition !

Instantaneous speed at $t = t_0$:





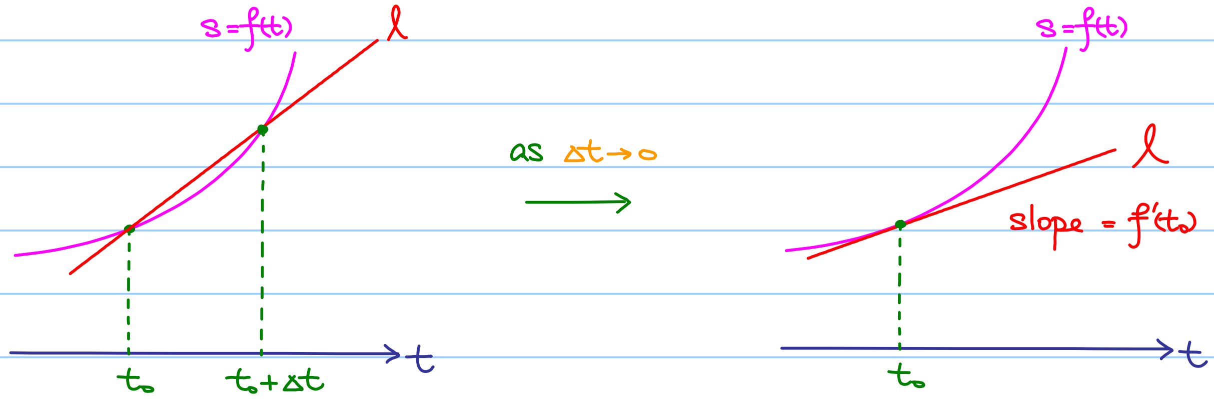
Average speed between t_0 and $t_0+\Delta t$

$$= \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{f(t_0+\Delta t) - f(t_0)}{\Delta t} = \text{slope of } \ell$$

Idea : Let Δt becomes smaller and smaller !

Instantaneous speed at $t=t_0$ is defined to be $\lim\limits_{\Delta t \to 0} \frac{f(t_0+\Delta t) - f(t_0)}{\Delta t}$

(provided it exists , if so , it is denoted by $f'(t_0)$)



Note : When $\Delta t \to 0$, ℓ becomes the tangent line at $t=t_0$, so

slope of the tangent line at $t=t_0$ $= f'(t_0)$

Example 5.1.1

If $s = f(t) = t^2$, find $f'(2)$ (instantaneous speed at $t=2$).

$$f'(2) = \lim_{\Delta t \to 0} \frac{f(2+\Delta t) - f(2)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(2+\Delta t)^2 - 2^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{2 \cdot 2 \Delta t + \Delta t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} 2 \cdot 2 + \Delta t = 2 \cdot 2 = 4$$

In general, we have $y = f(x)$, fix x_0.

Then $f'(x_0)$ means rate of change of y with respect to x at $x = x_0$.

Definition 5.1.1

$f(x)$ is said to be differentiable at $x = x_0$ if $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$ exists (called the first principle).

It is called the derivative of $f(x)$ at $x = x_0$ and it is denoted by $f'(x_0)$.

Note: By definition, if $f(x_0)$ is NOT well-defined, then $f'(x_0)$ is NOT well-defined.

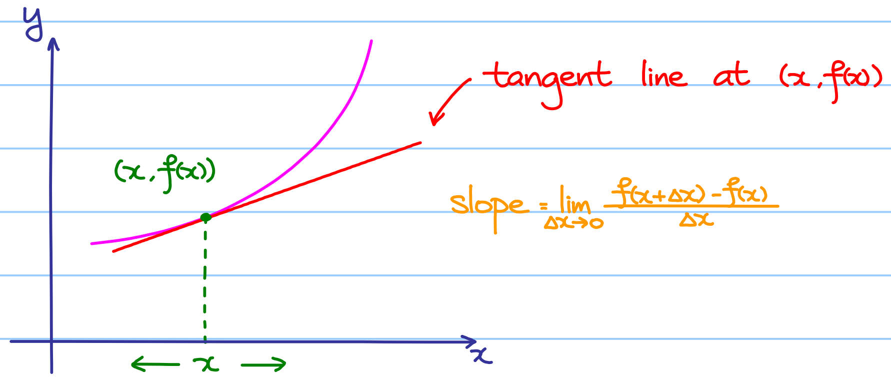
Let $\Delta x = x - x_0$, i.e. $x = x_0 + \Delta x$

When Δx tends to 0, x tends to x_0

Therefore, we have another formulation:

$f(x)$ is said to be differentiable at $x = x_0$ if $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists.

Perform the previous step at every point:



Recall : What is a function ?

Roughly speaking, given an input x, return a value

Now, we construct a new function, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ (if exists)

(i.e. given an input x, return the slope of the tangent line at $(x, f(x))$.)

Example 5.1.2

If $f(x) = x^2$, find $f'(x)$.

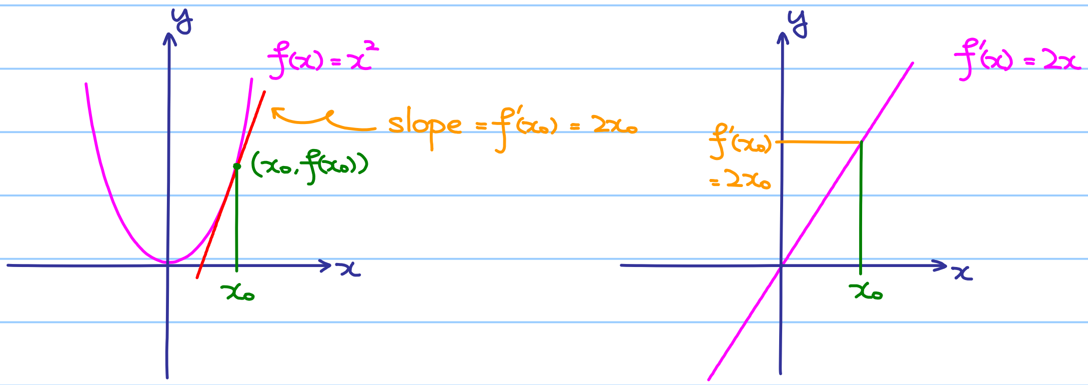
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

Relation between the graphs of $f(x) = x^2$ and $f'(x) = 2x$:



Notations :

$$y = f(x) = x^2$$

$$\frac{df}{dx} = \frac{dy}{dx} = f'(x) = 2x$$

$$\left.\frac{df}{dx}\right|_{x=3} = \left.\frac{dy}{dx}\right|_{x=3} = f'(3) = 2(3) = 6$$

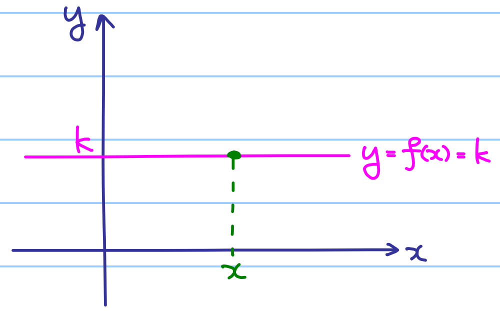
Definition 5.1.2

If $f : A \to \mathbb{R}$ is a function that is differentiable at every point in A,

then $f(x)$ is said to be a differentiable function.

Theorem 5.1.1

If $f(x) = k$, where k is a constant, then $f'(x) = 0$.



Note : tangent line at $(x, f(x))$ is horizontal

$\therefore f'(x) = 0$

Concrete computation :

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{k-k}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 0$$

$$= 0$$

Exercise 5.1.1

Find $f'(x)$ if

(a) $f(x) = x$ Ans : $f'(x) = 1$

(b) $f(x) = x^3$ $f'(x) = 3x^2$

Theorem 5.1.2

If $f(x) = x^r$, where r is a real number, then $f'(x) = rx^{r-1}$ whenever it is defined.

(Think: If $r = \frac{1}{2}$, $f(x) = \sqrt{x}$ which is defined when $x \geq 0$.)

proof:

We only prove the case $f(x) = x^n$, where n is a natural number.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(C_0^n x^n + C_1^n x^{n-1} \Delta x + C_2^n x^{n-2} \Delta x^2 + \cdots + C_n^n \Delta x^n) - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} C_1^n x^{n-1} + \underbrace{C_2^n x^{n-2} \Delta x + \cdots + C_n^n \Delta x^{n-1}}_{\text{terms with powers of } \Delta x}$$

$$= nx^{n-1}$$

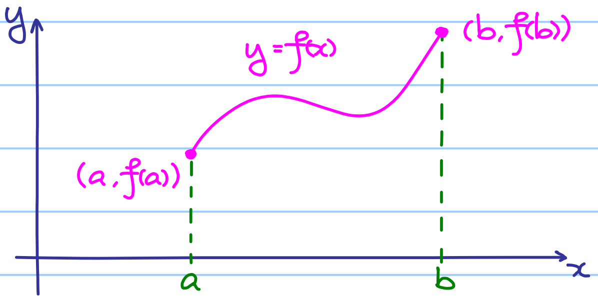
Differentiability on $[a,b]$

Definition 5.1.3

Let $f : [a,b] \longrightarrow \mathbb{R}$

f is said to be differentiable at $x=a$ if $\lim\limits_{\Delta x \to 0^+} \frac{f(a+\Delta x)-f(a)}{\Delta x}$ exists (or $\lim\limits_{x \to a^+} \frac{f(x)-f(a)}{x-a}$ exists).

f is said to be differentiable at $x=b$ if $\lim\limits_{\Delta x \to 0^-} \frac{f(b+\Delta x)-f(b)}{\Delta x}$ exists (or $\lim\limits_{x \to b^-} \frac{f(x)-f(b)}{x-b}$ exists).



Example 5.1.3

Let $f \cdot [0,1] \rightarrow \mathbb{R}$ defined by $f(x) = x^{\frac{3}{2}}$.

$$\lim_{\Delta x \to 0^+} \frac{f(0+\Delta x)-f(0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x^{\frac{3}{2}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0^+} \Delta x^{\frac{1}{2}}$$

$$= 0$$

.. f is differentiable at $x=0$ and $f'(0)=0$

5.2 Differentiability and Continuity

Theorem 5.2.1

If $f(x)$ is differentiable at $x=x_0$, then $f(x)$ is continuous at $x=x_0$

proof: By assumption, $\lim\limits_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ exists

Also, we know $\lim\limits_{\Delta x \to 0} \Delta x = 0$

$$\lim_{\Delta x \to 0} f(x_0+\Delta x) - f(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \cdot \Delta x$$

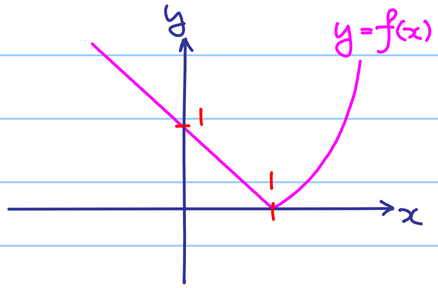
both exist

$$= \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta x \quad = f'(x_0) \cdot 0 = 0$$

$\therefore \lim\limits_{\Delta x \to 0} f(x_0+\Delta x) = f(x_0)$, so $f(x)$ is continuous at $x=x_0$.

However, the converse is NOT true.

Example 5.2.1



Let $f(x) = \begin{cases} x^2-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$

$$\lim_{\Delta x \to 0^+} \frac{f(1+\Delta x)-f(1)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{[(1+\Delta x)^2-1]-[1^2-1]}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{2\Delta x+\Delta x^2}{\Delta x} = 2$$

↑ (it means we are looking at small but positive Δx)

$$\lim_{\Delta x \to 0^-} \frac{f(1+\Delta x)-f(1)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{[1-(1+\Delta x)]-[1^2-1]}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$$

↑ (it means we are looking at small but negative Δx)

$$\lim_{\Delta x \to 0^+} \frac{f(1+\Delta x)-f(1)}{\Delta x} \neq \lim_{\Delta x \to 0^-} \frac{f(1+\Delta x)-f(1)}{\Delta x} \Rightarrow \lim_{\Delta x \to 0} \frac{f(1+\Delta x)-f(1)}{\Delta x} \text{ does NOT exist !}$$

∴ $f(x)$ is NOT differentiable at $x = 1$.

Exercise 5.2.1

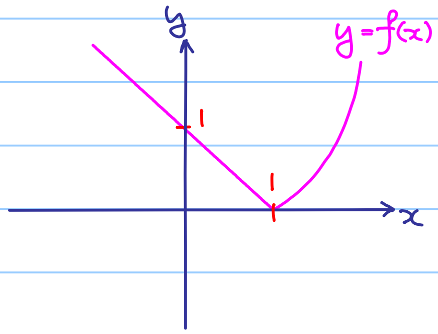
a) Show that $f(x)$ is continuous at $x=1$, i.e. $\lim_{x \to 1} f(x) = f(1)$.

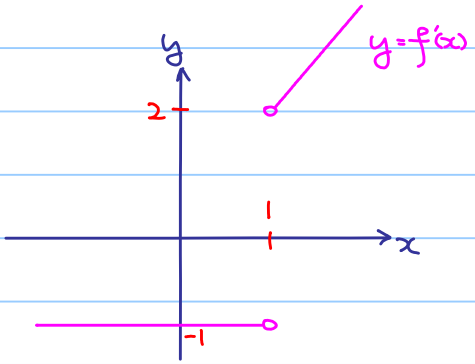
(Therefore, the converse statement of theorem 5.2.1 is NOT true.)

b) Write down $f'(x)$ for $x \neq 1$.

Answer :

$$f'(x) = \begin{cases} 2x & \text{if } x > 1 \\ \text{undefined} & \text{if } x = 1 \\ -1 & \text{if } x < 1 \end{cases}$$





5.3 Elementary Rules of Differentiation

Theorem 5.3.1

If $f(x)$ and $g(x)$ are differentiable functions, then

1) $(f+g)'(x) = f'(x) + g'(x)$

2) $(f-g)'(x) = f'(x) - g'(x)$

3) [product rule] $(f\cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

4) [qustient rule] $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ if $g(x) \neq 0$

proof of (3):

$$\lim_{\Delta x \to 0} \frac{(f\cdot g)(x+\Delta x) - (f\cdot g)(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x) + f(x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\underset{+}{g(x+\Delta x)} + f(x)\frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= f'(x)g(x) + f(x)g'(x)$$

$+$ $g(x)$ is differentiable

$\Rightarrow$ $g(x)$ is continuous

$\Rightarrow \lim_{\Delta x \to 0} g(x+\Delta x) = g(x)$

Direct consequence:

Theorem 5.3.2

If k is a constant and $f(x)$ is a differentiable function, then $(k\cdot f)'(x) = kf'(x)$.

proof:

Using the product rule and theorem 5.1.1

$$(k\cdot f)'(x) = \underbrace{(k)'}_{0}f(x) + kf'(x) = kf'(x)$$

Example 5.3.1

Find $\frac{d}{dx}(3x^2+7x-2)$

$$\frac{d}{dx}(3x^2+7x-2) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(7x) - \frac{d}{dx}(2)$$

$$= 3\frac{d}{dx}(x^2) + 7\frac{d}{dx}(x) - \frac{d}{dx}(2)$$

$$= 3(2x) + 7(1) - 0$$

$$= 6x + 7$$

Example 5.3.2

Find $\frac{d}{dx}(3x^2-5x+1)(2x+7)$

$$\frac{d}{dx}[(3x^2-5x+1)(2x+7)]$$
$$=\left[\frac{d}{dx}(3x^2-5x+1)\right](2x+7)+(3x^2-5x+1)\left[\frac{d}{dx}(2x+7)\right]$$
$$=(6x-5)(2x+7)+(3x^2-5x+1)(2)$$
$$=18x^2+22x-33$$

Try to compare : Expand $(3x^2-5x+1)(2x+7)$ and get $6x^3+11x^2-33x+7$
Then differentiate , get the same result ?

Example 5.3.3

Find the derivative of $\frac{2x}{x^2+1}$.

$$\frac{d}{dx}\frac{2x}{x^2+1}=\frac{[\frac{d}{dx}(2x)](x^2+1)-(2x)[\frac{d}{dx}(x^2+1)]}{(x^2+1)^2}$$
$$=\frac{2(x^2+1)-2x(2x)}{(x^2+1)^2}$$
$$=\frac{-2x^2+2}{(x^2+1)^2}$$

Example 5.3.4

Find the derivative of $\frac{1}{\sqrt{x}}+\sqrt{x}$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}+\sqrt{x}\right)=\frac{d}{dx}\left(x^{-\frac{1}{2}}+x^{\frac{1}{2}}\right)$$
$$=-\frac{1}{2}x^{-\frac{3}{2}}+\frac{1}{2}x^{-\frac{1}{2}}$$

5.4 Higher Derivatives

$s(t)$: distance functions (depends on time t)

(instantaneous) speed = rate of change of distance travelled with respect to t.

$v(t) = \frac{ds}{dt}$ (still a function of t)

Question : What is $\frac{dv}{dt}$?

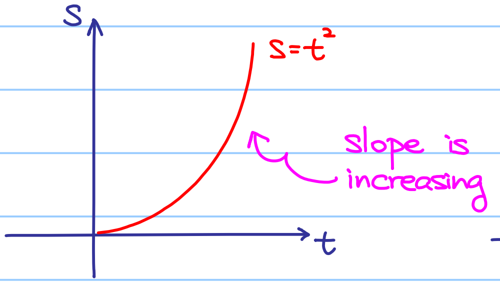
Answer : Acceleration !

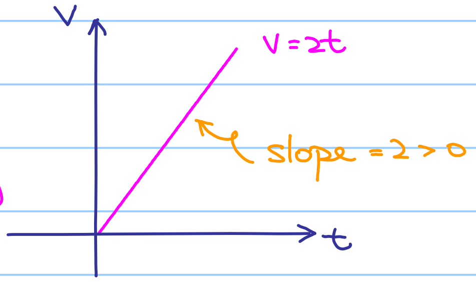
= rate of change of speed with respect to t.

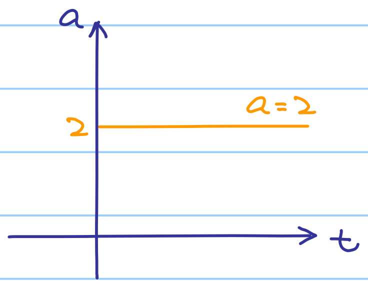
We write $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Example 5.4.1

$s(t) = t^2$ $\qquad$ $v(t) = \frac{ds}{dt} = 2t$ $\qquad$ $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 2$







speed is increasing

i.e. accelerating

Notations :

In general, let $y=f(x)$.

We have : (1 st derivative) $\frac{dy}{dx} = \frac{df}{dx} = f'(x)$

(2 nd derivative) $\frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = f''(x)$

(n th derivative) $\frac{d^ny}{dx^n} = \frac{d^nf}{dx^n} = f^{(n)}(x)$

5.5 Derivatives of Trigonometric Functions

Preparations :

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{2\sin^2(\frac{x}{2})}{x^2}$$

$$= \lim_{x\to 0}\frac{1}{2}\frac{\sin^2(\frac{x}{2})}{(\frac{x}{2})^2}$$

$$= \frac{1}{2}$$

Note: $\cos x = 1-2\sin^2(\frac{x}{2})$

$\therefore\ 1-\cos x = 2\sin^2(\frac{x}{2})$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot x$$

$$= \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\lim_{x\to 0}x$$

$$= \frac{1}{2}\cdot 0$$

$$= 0$$

$$\lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos x}{\Delta x} = \lim_{\Delta x\to 0}\frac{\cos x\cos\Delta x - \sin x\sin\Delta x - \cos x}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\cos x\frac{\cos\Delta x - 1}{\Delta x} - \sin x\frac{\sin\Delta x}{\Delta x}$$

$$= -\sin x$$

$(\because \lim_{\Delta x\to 0}\frac{\cos\Delta x - 1}{\Delta x} = 0$ and $\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x} = 1 .)$

$$\therefore\ \frac{d}{dx}\cos x = -\sin x$$

Exercise 5.5.1

Show that $\frac{d}{dx}\sin x = \cos x$ by using method similar to the above.

$\tan x = \frac{\sin x}{\cos x}$ $\qquad \sec x = \frac{1}{\cos x}$ $\qquad \csc x = \frac{1}{\sin x}$ $\qquad \cot x = \frac{\cos x}{\sin x}$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \overset{(*)}{\cdots} = \frac{1}{\cos^2 x} = \sec^2 x$$

(*) Exercise : By quotient rule

Exercise 5.5.2

Show that

a) $\frac{d}{dx}\sec x = \sec x\tan x$

b) $\frac{d}{dx}\csc x = -\csc x\cot x$

c) $\frac{d}{dx}\cot x = -\csc^2 x$

5.6 Derivative of e^x

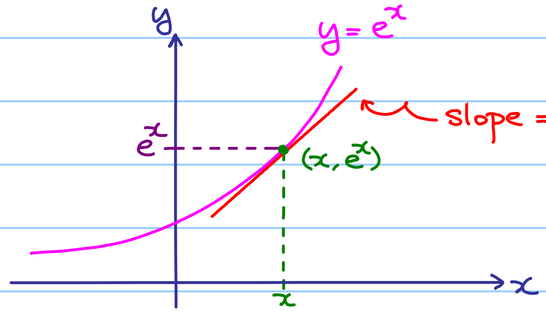
Cheating : $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\frac{d}{dx} e^x = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= e^x$$ (getting back itself)

Geometrical meaning :



Example 5.6.1

Find $\frac{d}{dx}[e^x(3x^2 + 7x - 2)]$

$$\frac{d}{dx}[e^x(3x^2+7x-2)] = [\frac{d}{dx} e^x](3x^2+7x-2) + e^x[\frac{d}{dx}(3x^2+7x-2)]$$

$$= e^x(3x^2+7x-2) + e^x(6x+7)$$

$$= e^x(3x^2+13x+5)$$

Question : How do we differentiate a more complicated function , such as $\sqrt{x^2+3x}$?

We need a tool called chain rule

5.7 Chain Rule

Theorem 5.7.1

If $f : B \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are differentiable functions such that $g(A) \subseteq B$,

then the composite function $(f \circ g) : A \to \mathbb{R}$ defined by $(f \circ g)(x) = f(g(x))$ is differentiable and

$(f \circ g)'(x) = f'(g(x))\, g'(x)$

Hard to understand ? Let's reformulate it as :

Let $u = g(x)$, $y = f(u) = f(g(x))$, then

the chain rule simply means $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Think : $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$

Example 5.7.1

Find the derivative of $\sqrt{x^2+3x}$.

Let $u = g(x) = x^2+3x$, $\qquad \frac{du}{dx} = 2x+3$

$y = f(u) = \sqrt{u}$ $\qquad \frac{dy}{du} = \frac{1}{2\sqrt{u}}$

then $f(g(x)) = \sqrt{x^2+3x}$

By the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= \frac{1}{2\sqrt{u}} \cdot (2x+3)$$

$$= \underbrace{\frac{1}{2\sqrt{x^2+3x}}}_{f'(g(x))} \cdot \underbrace{(2x+3)}_{g'(x)}$$

put $u = x^2+3x$ back

differentiate f
then put back $g(x)$

Example 5.7.2

Find the derivative of $(3x^2-2x)^{10}$

Let $u = g(x) = 3x^2-2x$ $\qquad \frac{du}{dx} = 6x-2$

$y = f(u) = u$ $\qquad \frac{dy}{du} = 10u^9$

then $y = f(g(x)) = (3x^2-2x)^{10}$

By chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= 10u^9 \cdot (6x-2)$$

$$= 10(3x^2-2x)^9 \cdot (6x-2)$$

put $u = 3x^2-2x$ back

$$= 20(3x^2-2x)^9 \cdot (3x-1)$$

Slogan: differentiate layer by layer.

Exercise 5.7.1

Show that $\frac{d}{dx} e^{ax} = ae^{ax}$

Show that $\frac{d}{dx} a^x = (\ln a) a^x$, for $a > 0$. (Hint: $a^x = e^{(\ln a^x)} = e^{(\ln a)x}$)

Exercise 5.7.2

Find the derivative of $\left(\frac{x}{x+1}\right)^2$

a) by using the chain rule;

b) by writing $\left(\frac{x}{x+1}\right)^2 = \frac{x^2}{(x+1)^2}$ and using the quotient rule.

Answer: Both equal to $\frac{2x}{(x+1)^3}$.

Example 5.7.3

Find the derivative of $e^{\sqrt{x^2+1}}$.

1st layer $y = e^w$ $\quad w = \sqrt{x^2+1}$

2nd layer $w = \sqrt{u}$ $\quad u = x^2+1$

3rd layer $u = x^2+1$

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx}$$

$$= e^{\sqrt{x^2+1}} \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

$$= \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$$

Example 5.7.3

Revisit of quotient rule :

$$\left(\frac{f}{g}\right)'(x) = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\,[g(x)]^{-1}\right)$$

$$= \frac{df}{dx}[g(x)]^{-1} + f(x)\frac{d}{dx}[g(x)]^{-1}$$

Apply the chain rule

$$= \frac{df}{dx}[g(x)]^{-1} + f(x)\left\{-[g(x)]^{-2}\frac{dg}{dx}\right\}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

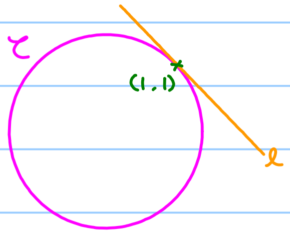
5.8 Implicit Differentiation

Example 5.8.1

$x^2+y^2=2$ —— $\mathcal{C}$

Locus of $\mathcal{C}$ is a circle centered at $(0,0)$ with radius $\sqrt{2}$.

Check: $(1,1)$ is a point lying on the circle.



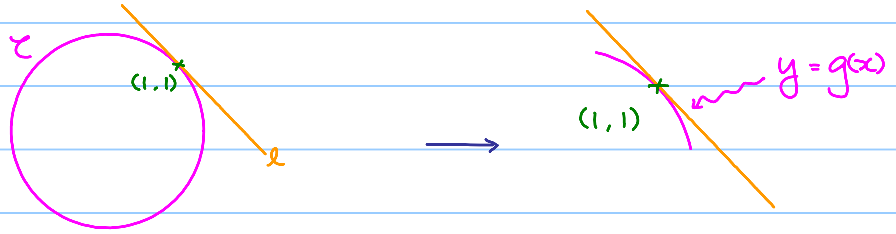
We want to find the equation of the tangent line ℓ

(i.e. need to know the slope of ℓ)

Note: $x^2+y^2=2$ is NOT a function!

Question: How to find $\frac{dy}{dx}$? (Actually, is it well defined ?)

Answer: Yes, roughly speaking.



The small segment of $\mathcal{C}$ containing $(1,1)$ can be regarded as the graph of some function $y=g(x)$. (In fact, $g(x)=\sqrt{2-x^2}$ in this case.)

How to find ? Do it as usual!

$$x^2+y^2=2$$

differentiate both sides with respect to x .

$$2x+\frac{d}{dx}y^2=0$$

$$2x+2y\frac{dy}{dx}=0 \qquad \text{(Applying chain rule)}$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

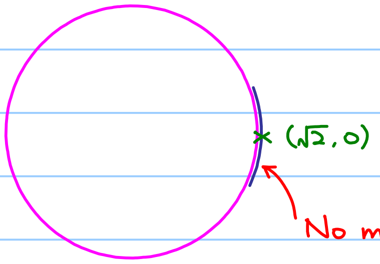
$\therefore \frac{dy}{dx}=-1$ when $(x,y)=(1,1)$.

We denote it by $\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)}=-1$

Remark.

$\frac{dy}{dx}$ is defined at a point of a curve only if a small arc containing the point can be regarded as the graph of some function $y=g(x)$.

$\therefore \frac{dy}{dx}$ is NOT defined when $(x,y) = (\pm\sqrt{2}, 0)$.



No matter how small the arc is,

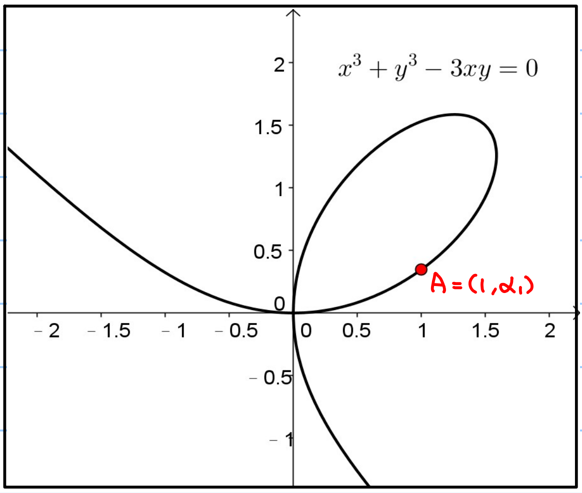
it cannot be realized as graph of some function $y=g(x)$.

Example 5.8.2

$$x^3+y^3-3xy=0 \quad \text{—} \quad \mathcal{C}$$

$$3x^2+3y^2\frac{dy}{dx}-3y-3x\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{y-x^2}{y^2-x}$$



If we want to find the slope of the tangent line at A.

putting $x=1$ into $\mathcal{C}$.

$$y^3-3y+1=0$$

NOT easy to solve!

FACT: The above equation has three roots, two roots α_1, α_2 are positive $(\alpha_1 < \alpha_2)$ one root is negative.

$A=(1,\alpha_1)$ and what we need is $\left.\frac{dy}{dx}\right|_{(x,y)=(1,\alpha_1)}$

Applications :

Example 5.8.3

Differentiation of Logarithmic Function

Let $y = \ln x$, $x > 0$. Then $e^y = x$,

differentiate both sides with respect to x .

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$\therefore \frac{d}{dx} \ln x = \frac{1}{x}$ for $x > 0$.

Exercise 5.8.1

By rewriting $\log_a x = \frac{\ln x}{\ln a}$, show that $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$.

Example 5.8 4

Let $y = \ln|x|$, $x \neq 0$. Find $\frac{dy}{dx}$.

We can rewrite $y = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$

For $x > 0$, we have just shown that $\frac{dy}{dx} = \frac{1}{x}$

For $x < 0$,

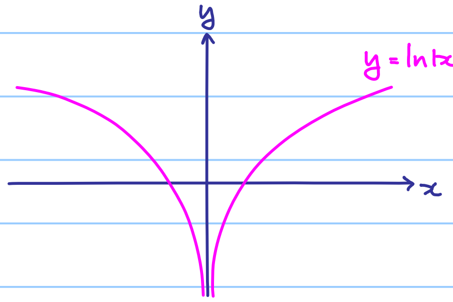
$$y = \ln(-x)$$

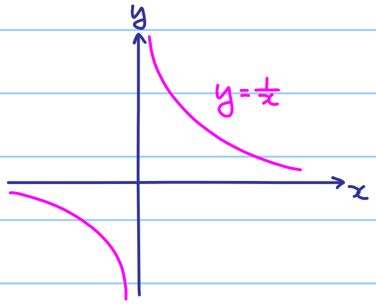
$$e^y = -x$$

$$e^y \frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = \frac{-1}{e^y} = \frac{1}{x}$$

$\therefore \frac{d}{dx} \ln|x| = \frac{1}{x}$ for $x \neq 0$.





Note: It is why $\int \frac{1}{x} dx = \ln|x| + C$.

Example 5.8.5

If $y=\sqrt[3]{\frac{(x-1)(x-2)^2}{x-4}}$, then find $\frac{dy}{dx}$.

Difficult to differentiate by using chain rule and quotient rule.

$$y^3 = \frac{(x-1)(x-2)^2}{x-4}$$

$$|y|^3 = \frac{|x-1||x-2|^2}{|x-4|}$$

$$\ln|y|^3 = \ln\frac{|x-1||x-2|^2}{|x-4|}$$

$$3\ln|y| = \ln|x-1| + 2\ln|x-2| - \ln|x-4|$$

$$\frac{3}{y}\frac{dy}{dx} = \frac{1}{x-1}+\frac{2}{x-2}-\frac{1}{x-4}$$

(Apply implicit differentiation)

$$\frac{dy}{dx} = \frac{y}{3}\left(\frac{1}{x-1}+\frac{2}{x-2}-\frac{1}{x-4}\right)$$

$$= \frac{1}{3}\sqrt[3]{\frac{(x-1)(x-2)^2}{x-4}}\left(\frac{1}{x-1}+\frac{2}{x-2}-\frac{1}{x-4}\right)$$

Example 5.8 6

Let $y=\frac{e^{5x}\sqrt[3]{x^2+1}}{(3x^2+1)^4}$ Find $\frac{dy}{dx}$.

$$y=\frac{e^{5x}\sqrt[3]{x^2+1}}{(3x^2+1)^4}$$

$$\ln y = 5x + \frac{1}{3}\ln(x^2+1) - 4\ln(3x^2+1)$$

Ex : ⋮

Ans : $\frac{dy}{dx} = \left[5 + \frac{2x}{3(x^2+1)} - \frac{24x}{3x^2+1}\right]\frac{e^{5x}\sqrt[3]{x^2+1}}{(3x^2+1)^4}$

Example 5.8 7

Differentiation of Inverse Trigonometric Functions

Let $y=\sin^{-1}x$, $\sin^{-1}: [-1,1] \longrightarrow [-\frac{\pi}{2},\frac{\pi}{2}]$. Then, $\sin y = x$.

differentiate both sides with respect to x.

$$\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$\sin y = x$, $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$

$$\cos y = \pm\sqrt{1-\sin^2 y}$$

$$= \sqrt{1-x^2} \quad \text{or} \quad -\sqrt{1-x^2}$$

(rejected , $-\frac{\pi}{2}\le y\le\frac{\pi}{2} \Rightarrow \cos y \ge 0$)

Let $y = \cos^{-1}x$, $\cos^{-1} : [-1,1] \longrightarrow [0,\pi]$. Then , $\cos y = x$.

differentiate both sides with respect to x .

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sin y}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\therefore \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

$\cos y = x$, $0 \le y \le \pi$

$\sin y = \pm\sqrt{1-\cos^2 y}$

$= \sqrt{1-x^2}$ or $-\sqrt{1-x^2}$

(rejected , $0 \le y \le \pi \Rightarrow \sin y \ge 0$)

Exercise 5.8.1

Let $y = \tan^{-1}x$, $\tan^{-1} : \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$.

Find $\frac{dy}{dx}$ Ans : $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

Example 5.8.8

Let $y = x^x$, $x > 0$. Find $\frac{dy}{dx}$.

Note : The power is NOT a constant, we cannot use the formular $\frac{d}{dx}x^n = nx^{n-1}$.

$$y = x^x$$

$$\ln y = \ln x^x = x\ln x$$

differentiate both sides with respect to x .

$$\frac{1}{y}\frac{dy}{dx} = \ln x + x \cdot \frac{1}{x}$$

$$= \ln x + 1$$

$$\frac{dy}{dx} = (\ln x + 1)y = (\ln x + 1)x^x$$

Example 5.8.9

Suppose $x^3 + y^3 - 3xy = 0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$x^3 + y^3 - 3xy = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y-x^2}{y^2-x}$$

differentiate both sides with respect to x again

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}(y^2-x) - (y-x^2)(2y\frac{dy}{dx} - 1)}{(y^2-x)^2}$$

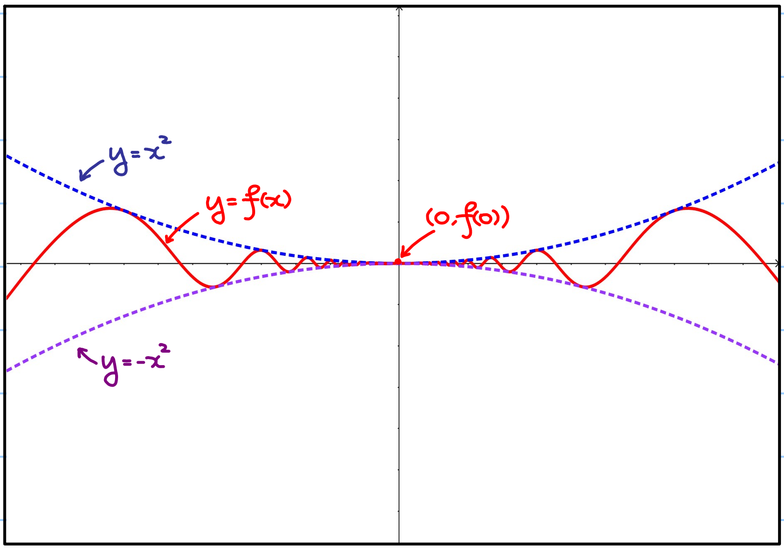
Sub. $\frac{dy}{dx} = \frac{y-x^2}{y^2-x}$ back to express $\frac{d^2y}{dx^2}$ in terms of x and y only , if you want. (Nightmare !)

5.9 More on Differentiability

Example 5.9.1

Let $f(x) = \begin{cases} x^2 \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Does $f'(0)$ exist ?



$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x)-f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x^2 \cos\frac{1}{\Delta x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x \cos\frac{1}{\Delta x}$$

$$= 0$$

By sandwich theorem

If $x \neq 0$, $f'(x) = 2x\cos\frac{1}{x} + x^2(-\sin\frac{1}{x})(-\frac{1}{x^2})$

$$= 2x\cos\frac{1}{x} + \sin\frac{1}{x}$$

$\therefore$ f is a differentiable function, i.e. differentiable at every point.

Note : It is wrong to say $f'(x) = 2x\cos\frac{1}{x} + \sin\frac{1}{x}$, so $f'(0)$ does NOT exist

Now, $f'(x) = \begin{cases} 2x\cos\frac{1}{x} + \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

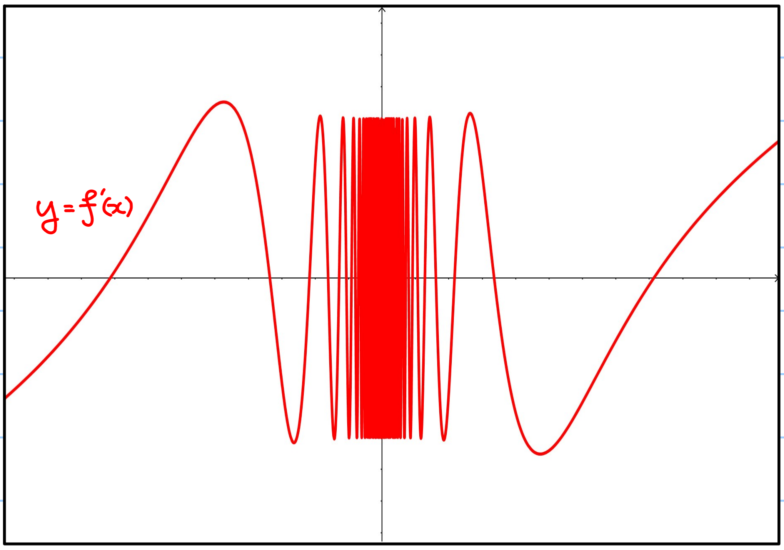
Exercise :

Show $\lim\limits_{x \to 0} f'(x)$ does NOT exist

($\Rightarrow$ $f'(x)$ is NOT continuous at $x=0$)

$\therefore$ f is differentiable ("good" in some sense)

but $f'(x)$ can be "bad".



Example 5.9.2

Let $f:\mathbb{R}\to\mathbb{R}$ be a non-constant function such that

(i) f is differentiable at some $x_0\in\mathbb{R}$

(ii) $f(x+y)=f(x)f(y)$ for all $x,y\in\mathbb{R}$.

Show that :

a) $f(x)\neq 0$ for all $x\in\mathbb{R}$ and $f(0)=1$.

b) f is differentiable at every $x\in\mathbb{R}$ and $f'(x)=\frac{f'(x_0)}{f(x_0)}f(x)$.

a) If $f(a)=0$ for some $a\in\mathbb{R}$

then for any $x\in\mathbb{R}$, we have

$$f(x)=f(x-a+a)=f(x-a)f(a)=0$$

i.e. $f(x)$ is constant zero (Contradict to the assumption)

$\therefore f(x)\neq 0 \quad \forall x\in\mathbb{R}$.

Putting $x=y=0$,

$$f(0+0)=f(0)f(0)$$

$$f(0)=[f(0)]^2$$

$$f(0)=1 \quad \text{or} \quad 0 \ (\text{rejected})$$

b) f is differentiable at x_0

$$\Rightarrow f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{f(x_0)f(\Delta x)-f(x_0)}{\Delta x}$$

$$\therefore \lim_{\Delta x\to 0}\frac{f(\Delta x)-1}{\Delta x}=\frac{f'(x_0)}{f(x_0)} \qquad (\because f(x_0)\neq 0)$$

Now, $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$

$$=\lim_{\Delta x\to 0}\frac{f(x)f(\Delta x)-f(x)}{\Delta x}$$

$$=f(x)\cdot\lim_{\Delta x\to 0}\frac{f(\Delta x)-1}{\Delta x}$$

$$=\frac{f'(x_0)}{f(x_0)}f(x)$$

$\therefore f$ is differentiable at every $x\in\mathbb{R}$ and $f'(x)=\frac{f'(x_0)}{f(x_0)}f(x)$.

(In fact, $f(x)=e^{kx}$ for some non-zero constant k.)

Exercise 5.9.1

Let f be a differentiable function such that

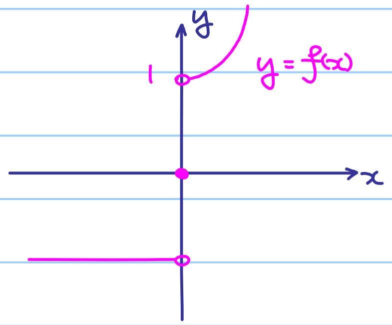
$$f(x+y) = f(x) + f(y) + 3xy(x+y) \quad \forall\, x, y \in \mathbb{R}.$$

a) Show that $f'(0) = \lim_{\Delta x \to 0} \frac{f(\Delta x)}{\Delta x}$.

b) Hence, show that $f'(x) = f'(0) + 3x^2$. (In fact, $f(x) = C + f'(0)x + x^3$ if you know integration.)

Exercise 5.9.2

Let $f(x) = \begin{cases} x^2+1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$



(a) Write down $f'(x)$ explicitly

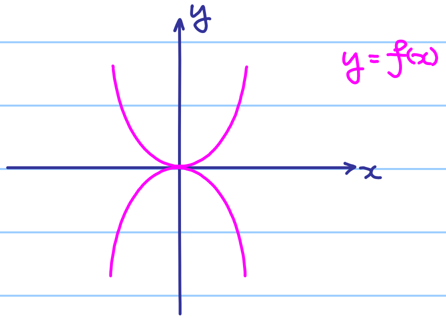
(b) Show that f is not differentiable at $x=0$.

(c) Show that $\lim_{x\to 0^-} f'(x) = 0$ and $\lim_{x\to 0^+} f'(x) = 0$, so $\lim_{x\to 0} f'(x) = 0$.

Therefore, $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x)$ is insufficient to show f is differentiable at $x=0$.

Exercise 5.9.3

Let $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ -x^2 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$



Show that $f(x)$ is differentiable only at $x=0$ and $f'(0)=0$.

Therefore, $f'(0)$ exists while $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$ are not.

Summary:

$$\begin{tikzcd}
 & & f' \text{ is differentiable at } x=x_0 \\
f \text{ is differentiable at } x=x_0 \arrow[rru, phantom, "\not\Rightarrow"] \arrow[rr, phantom, "\not\Rightarrow"] \arrow[rrd, phantom, "\not\Rightarrow", "\text{Example 5.9.1}"'] \arrow[dd, Rightarrow, shift right=2, "\text{Theorem 5.2.1}"'] & & f' \text{ is continuous at } x=x_0 \\
 & & \lim_{x\to x_0} f'(x) \text{ exists} \\
f \text{ is continuous at } x=x_0 \arrow[uu, Rightarrow, shift right=2, "/" marking, "\text{Example 5.2.1}"'] \arrow[d, Rightarrow, shift right=2, "\text{Definition 4.1.1}"'] & & \\
\lim_{x\to x_0} f(x) \text{ exists} \arrow[u, Rightarrow, shift right=2, "/" marking, "\text{Example 3.1.3}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
f \text{ is differentiable at } x=x_0 \arrow[r, Rightarrow, shift left=2, "/" marking, "\text{Exercise 5.9 3}"] & \lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) \arrow[l, Rightarrow, shift left=2, "/" marking, "\text{Exercise 5.9.2}"]
\end{tikzcd}$$

§ 6 Applications of Differentiation

6.1 Rolle's Theorem and Mean Value Theorem

Theorem 6.1.1

Let $f:(a,b)\to\mathbb{R}$ be a function and $c\in(a,b)$ such that

1) $f'(c)$ exists

2) f attains maximum (or minimum) at $x=c$.

Then, we have $f'(c)=0$.

proof: Assume f attains maximum at $x=c$

$$f'(c) \text{ exist } \Rightarrow \lim_{\Delta x\to 0^+}\frac{f(c+\Delta x)-f(c)}{\Delta x} = \lim_{\Delta x\to 0^-}\frac{f(c+\Delta x)-f(c)}{\Delta x} = f'(c)$$

By theorem 3.2.2 : $\frac{f(c+\Delta x)-f(c)}{\Delta x}\leqslant 0$ for all $\Delta x>0 \Rightarrow f'(c)=\lim_{\Delta x\to 0^+}\frac{f(c+\Delta x)-f(c)}{\Delta x}\leqslant 0$

$\frac{f(c+\Delta x)-f(c)}{\Delta x}\geqslant 0$ for all $\Delta x<0 \Rightarrow f'(c)=\lim_{\Delta x\to 0^+}\frac{f(c+\Delta x)-f(c)}{\Delta x}\geqslant 0$

$\therefore f'(c)=0$.

Theorem 6.1.2 (Rolle's Theorem)

Let $F:[a,b]\to\mathbb{R}$ be a function such that

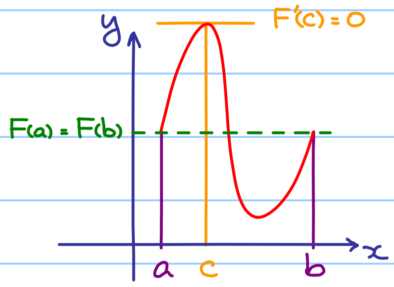
1) F is continuous on $[a,b]$

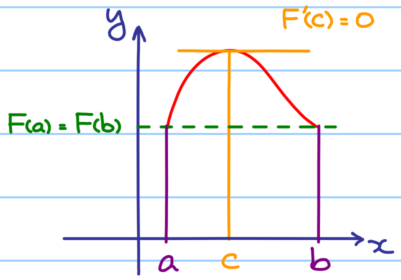
2) F is differentiable on (a,b)

3) $F(a)=F(b)$

then there exists $c\in(a,b)$ such that $F'(c)=0$.

Geometrical meaning:





Idea of proof:

By the Maximum-Minimum Theorem, there exist $x_m, x_M \in [a,b]$ such that $F(x_m) \le F(x) \le F(x_M)$ for all $x \in [a,b]$.

Case 1: Either x_m or x_M lies on (a,b)

then $F'(x_m) = 0$ or $F'(x_M) = 0$ (By theorem 6.1.1)

Case 2: Both x_m and x_M lies on boundary points of $[a,b]$,

By assumption, $F(a) = F(b)$ which forces that $F(x)$ is constant on $[a,b]$

so $f'(x) = 0$ for all $x \in (a,b)$

Theorem 6.1.3 (Mean Value Theorem)

Let $f : [a,b] \to \mathbb{R}$ be a function such that

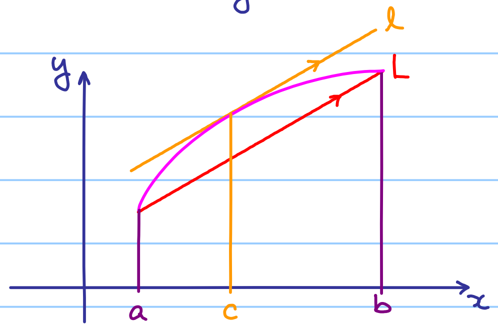
1) f is continuous on $[a,b]$

2) f is differentiable on (a,b)

then there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

slope of ℓ slope of L.

Geometrical meaning:



Another interpretation:

$$\underbrace{f(b) - f(a)}_{\text{change of } f} = f'(c)\,\underbrace{(b-a)}_{\text{change of } x}$$

When x changes from a to b, change of $x = b-a$ and change of $f = f(b) - f(a)$, and they are related by the derivative f' at some point $c \in (a,b)$.

Therefore, if we have a control on f', then we obtain a control on change of f.

Idea of proof:

Looking for $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

i.e. looking for a solution in (a,b) of the equation

$$\underbrace{f'(x) - \frac{f(b)-f(a)}{b-a}}_{\uparrow} = 0 .$$

Idea: Realize this as $F'(x)$ and apply Rolle's theorem.

proof:

Let $F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$

Check: 1) F is continuous on $[a,b]$

2) F is differentiable on (a,b)

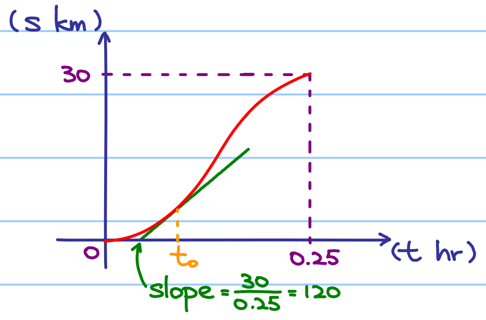
3) $F(a) = F(b) = 0$

Apply Rolle's Theorem to F, the result follows.

Question:

A vehicle is speeding on a highway if its speed ≥ 120 km/hr (at some moment)

If the length of the highway is 30 km and if a driver only spent 15 minutes on the highway. Should he be arrested?



By the MVT, there exists $t_0 \in (0, 0.25)$

such that slope of the tangent at $t = t_0$ $= \frac{30}{0.25} = 120$

i.e. instantaneous speed at $t = t_0$ $= 120$ km/hr

6.2 Applications of Mean Value Theorem

Example 6.2.1

We know $\ln 1 = 0$, how about $\ln 1.1$?

Let $f(x) = \ln x$, for $x > 0$, which is a differentiable function.

In particular, f is continuous on $[1, 1.1]$,

f is differentiable on $(1, 1.1)$.

Apply MVT, $\exists\, c \in (1, 1.1)$ such that $f(1.1) - f(1) = f'(c)\,(1.1 - 1)$

$$\ln 1.1 - \ln 1 = \frac{1}{c} \cdot (0.1)$$

$$\ln 1.1 = \frac{1}{10} \cdot \frac{1}{c}$$

Note that $1 < c < 1.1 = \frac{11}{10}$, so $\frac{10}{11} < \frac{1}{c} < 1$ (Control of f')

$$\frac{1}{11} < \frac{1}{10} \cdot \frac{1}{c} < \frac{1}{10}$$

$$\therefore\ 0.909 \approx \frac{1}{11} < \ln 1.1 < \frac{1}{10} = 0.1$$

Theorem 6.2.1

If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable and $f'(x) = 0 \quad \forall x \in \mathbb{R}$,

then $f(x)$ is a constant function

proof: Fix $x_0 \in \mathbb{R}$, let $x \in \mathbb{R} \setminus \{x_0\}$

If $x > x_0$, note f is differentiable everywhere (in particular, on (x_0, x))

$\Rightarrow f$ is continuous everywhere (in particular, on $[x_0, x]$)

Apply MVT, $\exists\, c \in (x_0, x)$ such that

$$f(x_0) - f(x) = \underset{\substack{\| \\ 0}}{\underline{f'(c)}}\,(x - x_0) = 0$$

by assumption.

i.e. $f(x) = f(x_0) \quad \forall x > x_0$

We have similar result if $x < x_0$, the result follows.

Example 6.2.2

Let $f(x) = \cos^2 x + \sin^2 x$

$f'(x) = -2\cos x \sin x + 2 \sin x \cos x = 0$

$\therefore\ \cos^2 x + \sin^2 x$ is a constant.

In particular, $f(0) = 1$, so $f(x) = \cos^2 x + \sin^2 x = 1$

Theorem 6.2.2

If $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable functions such that $f'(x) = g'(x)$ for all $x \in \mathbb{R}$,
then $f(x) = g(x) + C$, where C is a constant.

proof: Let $h(x) = f(x) - g(x)$

Then $h'(x) = f'(x) - g'(x) = 0$

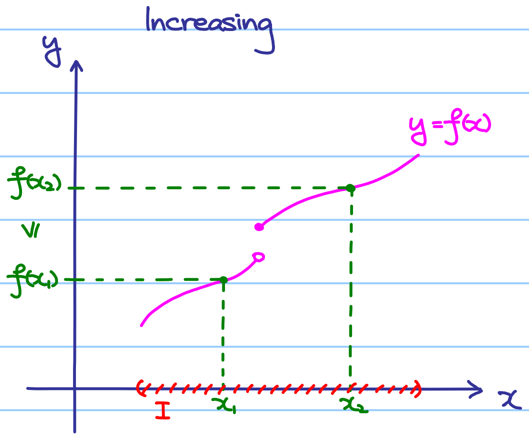
$\therefore h(x) = C$, where C is a constant. i.e. $f(x) = g(x) + C$.

Next, we are going to discuss how differentiation helps to find maximum / minimum points of a function.
Firstly, we make some preparations:

6.3 Increasing / Decreasing Functions

Definition 6.3.1

Let I be an interval and let $f : I \to \mathbb{R}$ be a function such that
$f(x_1) \le f(x_2)$ $\quad$ ($f(x_1) \ge f(x_2)$) for all $x_1 < x_2$.
then $f(x)$ is called an increasing (a decreasing) function.†



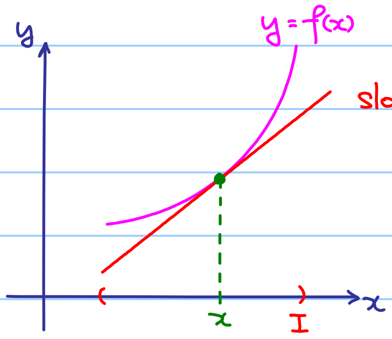
Roughly speaking:
The larger x we input
the larger y we get!

† If we have a strictly inequality, it is called a strictly increasing (decreasing) function.

Theorem 6.3.1

Let I be an interval and let $f: I \to \mathbb{R}$ be a differentiable function.

If $f'(x) \geq 0$ †† $(f'(x) \leq 0)$ for all $x \in I$ then f is increasing (decreasing) on I.



slope = $f'(x) \geq 0$ ⇒ Increasing

†† If we have strict inequality, then $f(x)$ is strictly increasing (decreasing) on (a,b).

proof :

If $x_1, x_2 \in I$ with $x_1 < x_2$, then f is continuous on $[x_1, x_2]$

f is differentiable on (x_1, x_2).

Applying MVT to f on $[x_1, x_2]$,

$$\exists\, c \in (x_1, x_2) \text{ such that } f(x_2) - f(x_1) = \underbrace{f'(c)}_{\substack{\geq\\ 0}} \underbrace{(x_2 - x_1)}_{\substack{>\\ 0}} \geq 0$$

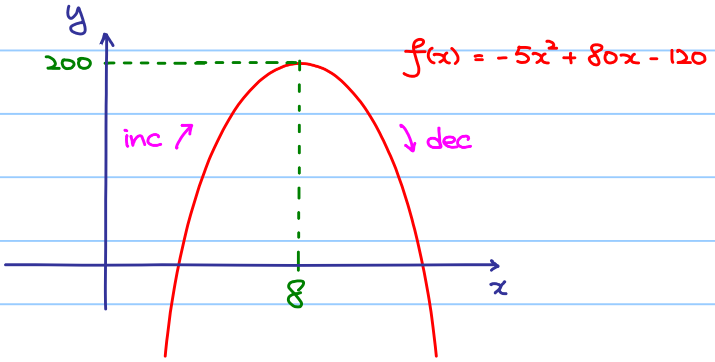
By assumption

Example 6.3.1

$f(x) = -5x^2 + 80x - 120$

$f'(x) = -10x + 80$

$f'(x) > 0$	$f'(x) < 0$
$-10x + 80 > 0$	$-10x + 80 < 0$
$x < 8$	$x > 8$



∴ $f(x)$ is strictly increasing when $x < 8$ and

$f(x)$ is strictly decreasing when $x > 8$.

Not hard to understand why $f(x)$ attains maximum when $x = 8$

and maximum value $= f(8) = 200$

Note : $f'(8) = 0$

Remark : Verify the answer by using completing square.

Question :

1) If $f'(x) > 0$ for $x < a$ and $f'(x) < 0$ for $x > a$,

is it enough to say that f attains maximum at $x = a$?

2) If we want to find all extrema,

is it enough to solve $f'(x) = 0$?

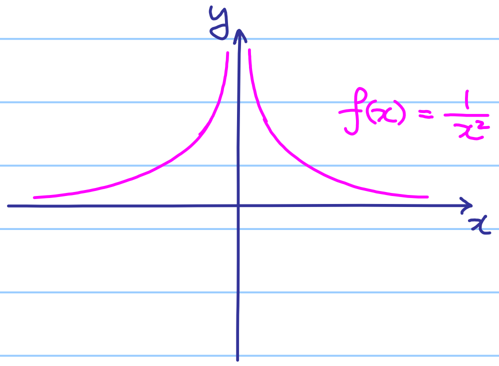
Example 6.3.2

Let $f(x) = \frac{1}{x^2}$, $x \neq 0$.

$f'(x) = -\frac{2}{x^3}$

$f'(x) > 0$ for $x < 0$

$f'(x) < 0$ for $x > 0$



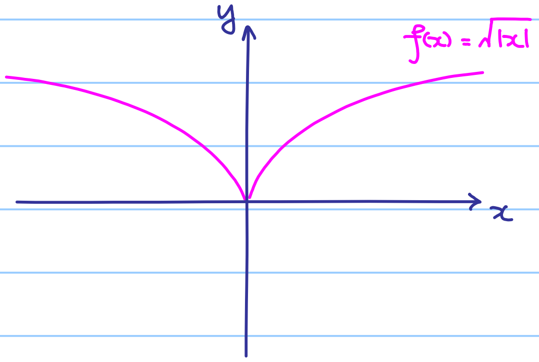
$\therefore$ $f(x)$ is strictly increasing when $x < 0$

$f(x)$ is strictly decreasing when $x > 0$

However, $f(0)$ is NOT well-defined, so there is NO maximum point.

Example 6.3.3

Let $f(x) = \sqrt{|x|}$



Rewrite :

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases}$$

If $x > 0$, $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}} > 0$

If $x < 0$, $f(x) = \sqrt{-x}$, then $f'(x) = -\frac{1}{2\sqrt{-x}} < 0$

$\therefore$ $f(x)$ is strictly increasing when $x > 0$

$f(x)$ is strictly decreasing when $x < 0$

However, $\lim\limits_{\Delta x \to 0^+} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim\limits_{\Delta x \to 0^+} \frac{\sqrt{\Delta x}}{\Delta x} = \lim\limits_{\Delta x \to 0^+} \frac{1}{\sqrt{\Delta x}}$ which does NOT exist ,

$\Rightarrow$ $\lim\limits_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$ does NOT exist

$\Rightarrow$ $f'(0)$ does NOT exist

but as we can see f still attains minimum at $x = 0$.

$\therefore$ Solving $f(x) = 0$ to find max / min is NOT enough

However, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \sqrt{x} = 0$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \sqrt{-x} = 0$

$f(0) = 0$

$\therefore$ $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0) = 0$ and so f is continuous at $x = 0$

By the first derivative check , $f(x)$ attains minimum at $x = 0$.

Answers for both questions 1 and 2 are negative ,

so , what is the exact statement of finding an extrema ?

6.4 First Derivative Check

Theorem 6.4.1 (First Derivative Check)

Let I be an open interval and let $a \in I$.

Let $f : I \to \mathbb{R}$ be a function such that

1) f is continuous

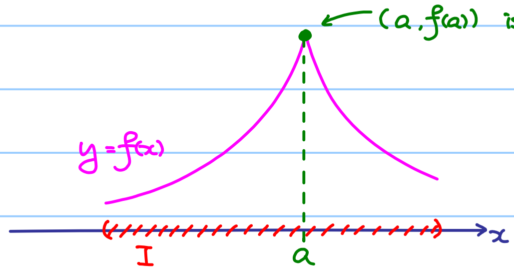
2) $f'(x) \geq 0$ $(f'(x) \leq 0)$ for all $x \in I$ with $x < a$

3) $f'(x) \leq 0$ $(f'(x) \geq 0)$ for all $x \in I$ with $x > a$

Then $(a, f(a))$ is a relative maximum (minimum).

Note : We do NOT require the differentiability of f at $x = a$, but only the continuity of f at $x = a$.

Geometrical meaning :



Remember the slogan : Change of sign of $f'(x)$ at $x = a$

proof:

Let $x \in I$ and $x \in a$.

Note : f is continuous on $[x, a]$ and

f is differentiable on (x, a)

apply the MVT, there exists $c \in (x, a)$ such that

$$f(a) - f(x) = \underbrace{f'(c)}_{\geq 0} \underbrace{(a - x)}_{> 0} \geq 0$$

By assumption

$\therefore f(x) \leq f(a)$ for all $x \in I$ with $x < a$

Similarly, we can also show that $f(x) \leq f(a)$ for all $x \in I$ with $x > a$

$\therefore f(x) \leq f(a)$ for all $x \in I$, i.e. $(a, f(a))$ is a relative maximum.

Example 6.4.1

Prove that $e^x \geqslant 1+x$ (i.e. $e^x - x - 1 \geqslant 0$) for all $x \in \mathbb{R}$.

Let $f(x) = e^x - x - 1$

(Want to find the global minimum of $f(x)$ and see if it is $\geqslant 0$.)

$f'(x) = e^x - 1$

$f'(x) > 0$ if $x > 0$ and $f'(x) < 0$ if $x < 0$

f is strictly increasing when $x > 0$ and strictly decreasing when $x < 0$

(and f is continuous at $x = 0$.)

f attains minimum when $x = 0$ (By 1st derivative check)

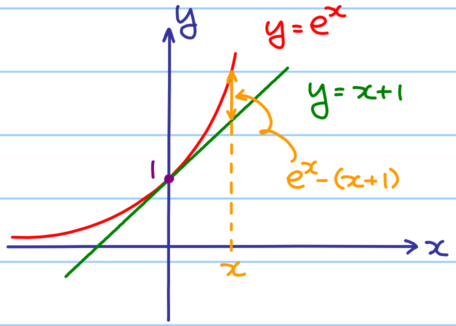
(In fact, global minimum, why?)

$\therefore f(x) \geqslant f(0) \quad \forall x \in \mathbb{R}$ ——— (*)

$= e^0 - 0 - 1$

$= 0$

Note: The equality holds iff $x = 0$



Definition 6.4.1

If $f'(a) = 0$, then $(a, f(a))$ is said to be a stationary point.

However, a stationary point is NOT necessary to be a relative extrema

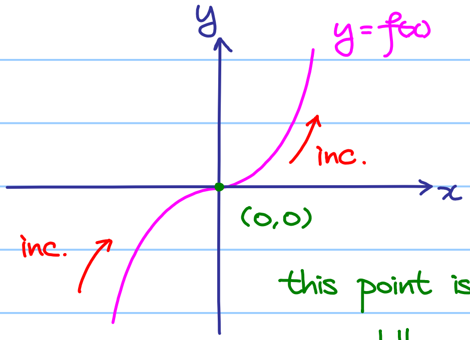
Example 6.4.2

If $f(x) = x^3$, then $f'(x) = 3x^2$

Note: 1) $f'(0) = 0$

2) $f'(x) = 3x^2 > 0$ for $x \neq 0$

i.e. No change of sign of $f'(x)$ at $x = 0$.



this point is called a saddle point

Note: a stationary point is NOT necessary to be a max./min. point!

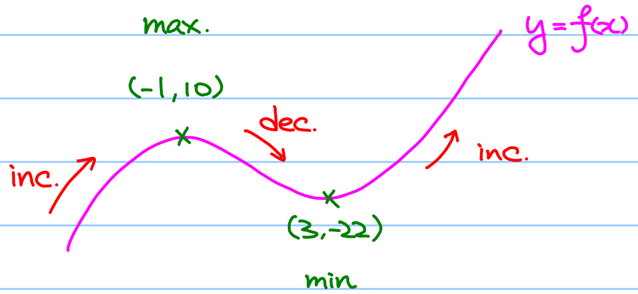
Example 6.4.3

If $f(x) = x^3 - 3x^2 - 9x + 5$

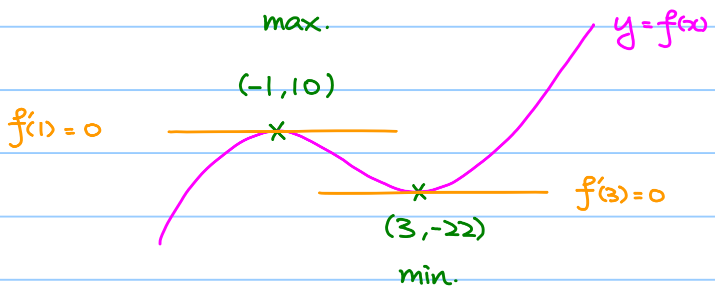
then $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$

$f'(x) > 0$ if $x > 3$ or $x < -1$

$f'(x) < 0$ if $-1 < x < 3$



Furthermore,



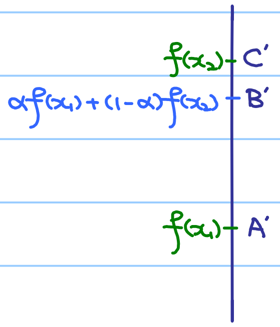
6.5 Second Derivative Check

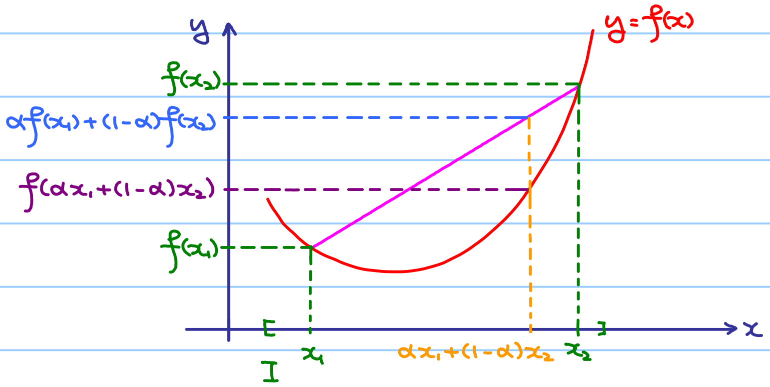
Definition 6.5.1

Let I be an interval and let $f: I \to \mathbb{R}$ be a function.

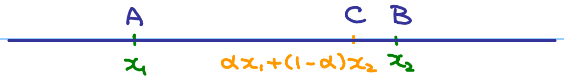
f is concave up on I if $\forall x_1, x_2 \in I$ with $x_1 < x_2$, $\alpha \in (0,1)$, we have

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$





$A'C' . C'B' = (1-\alpha) : \alpha$



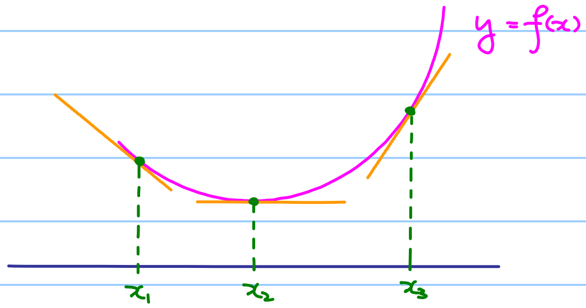
$AC : CB = (1-\alpha) : \alpha$

Similarly, f is concave down on I if $\forall x_1, x_2 \in I$ with $x_1 < x_2$, $\alpha \in (0,1)$, we have

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha) f(x_2)$$

Let I be an interval. $f''(x) > 0$ for $x \in I \Rightarrow f'(x)$ is strictly increasing.

Geometrical meaning:



Slope of the tangent line at $(x, f(x))$ increases as x increases!
(NOT $f(x)$ is increasing!)

Theorem 6.5.1

Let I be an interval.

If $f''(x) \geq 0$ ($f''(x) \leq 0$) $\forall x \in I$, then $f(x)$ is concave up (down) on I.

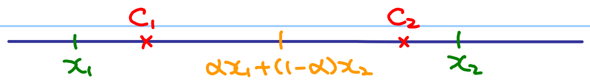
proof:

Suppose that $x_1, x_2 \in I$ with $x_1 < x_2$, $\alpha \in (0,1)$, then $x_1 < \alpha x_1 + (1-\alpha)x_2 < x_2$

Applying MVT to f on $[x_1, \alpha x_1 + (1-\alpha)x_2]$ and $[\alpha x_1 + (1-\alpha)x_2, x_2]$,

$\exists c_1 \in (x_1, \alpha x_1 + (1-\alpha)x_2)$ and $\exists c_2 \in (\alpha x_1 + (1-\alpha)x_2, x_2)$ s.t.

$$\frac{f(\alpha x_1 + (1-\alpha)x_2) - f(x_1)}{(\alpha x_1 + (1-\alpha)x_2) - x_1} = f'(c_1) \quad \text{and} \quad \frac{f(x_2) - f(\alpha x_1 + (1-\alpha)x_2)}{x_2 - (\alpha x_1 + (1-\alpha)x_2)} = f'(c_2)$$



Note than $f''(x) \geq 0$ on I, f' is increasing on I and so $f'(c_1) \leq f'(c_2)$.

$$\frac{f(\alpha x_1 + (1-\alpha)x_2) - f(x_1)}{(\alpha x_1 + (1-\alpha)x_2) - x_1} \leq \frac{f(x_2) - f(\alpha x_1 + (1-\alpha)x_2)}{x_2 - (\alpha x_1 + (1-\alpha)x_2)}$$

$$(x_2 - x_1) f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha(x_2 - x_1) f(x_1) + (1-\alpha)(x_2 - x_1) f(x_2)$$

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$

Theorem 6.5.2 (Second Derivative Check)

Let I be an open interval and let $a \in I$

If $f : I \to \mathbb{R}$ be a function such that

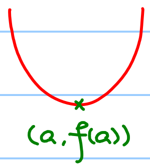
1) $f'(a) = 0$ (i.e. $(a, f(a))$ is a stationary point.)

2) $f''(a) < 0$ ($f''(a) > 0$)

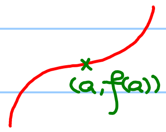
then $(a, f(a))$ is a relative maximum (minimum).

Idea. If $(a, f(a))$ is a stationary point, we have 4 possible cases:

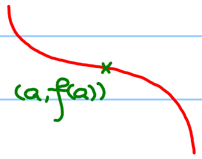
① Min.



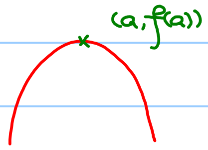
② Saddle point



③ Saddle point



④ Max.



Roughly speaking: $f''(a) < 0 \Rightarrow f$ is concave down around $x = a$

$\therefore$ Case ①, ②, ③ are ruled out, and so f attains maximum at $x = a$.

proof.

Given that $\lim_{x \to a} \frac{f'(x) - f'(a)}{x - a} = f''(a) > 0$, and $f'(a) = 0$

$\lim_{x \to a^+} \frac{f'(x)}{x - a} = f''(a) < 0$, so $f'(x) < 0$ ($\Rightarrow f$ is strictly dec.) if x is slightly greater than a.

positive

$\lim_{x \to a^-} \frac{f'(x)}{x - a} = f''(a) < 0$, so $f'(x) > 0$ ($\Rightarrow f$ is strictly inc.) if x is slightly smaller than a.

negative

The result follows by the first derivative check.

Caution: If $f''(a) = 0$, then NO conclusion!

Consider $f(x) = x^4, x^3, -x^4$

We have $f'(0) = f''(0) = 0$ in each case, but $(0,0)$ is

- min. for the 1st case
- saddle point for the 2nd case.
- max. for the 3rd case

Example 6.5 1

If $f(x) = x^3 - 3x^2 - 9x + 5$

then $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$

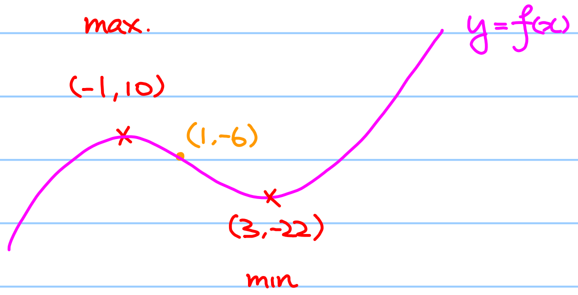
$f'(x) > 0$ if $x > 3$ or $x < -1$

$f'(x) < 0$ if $-1 < x < 3$

$f''(x) = 6x - 6$

$f''(x) > 0$ if $x > 1$ $\qquad f''(-1) = 12 < 0$

$f''(x) < 0$ if $x < 1$ $\qquad f''(3) = 12 > 0$



$f'(x)$	+ve	-1	-ve	3	+ve
$f(x)$	inc.		dec		inc.

$f''(x)$	-ve	1	+ve
$f(x)$	convex		cancave

Note: The curve changes from being convex to concave at $(1,6)$.

This point is called a point of inflection.

Definition 6.5.1

Let I be an open interval and let $a \in I$

Let $f : I \to \mathbb{R}$ be a function such that

1) f is continuous

2) $f''(x) > 0$ $\quad (f''(x) < 0)$ for all $x \in I$ with $x < a$

3) $f''(x) < 0$ $\quad (f''(x) > 0)$ for all $x \in I$ with $x > a$

then $(a, f(a))$ is said to be a point of inflection.

Remember the slogan: Change of sign of $f''(x)$ at $x = a$

Example 6.5.2

$f(x) = 12x^5 - 105x^4 + 340x^3 - 510x^2 + 360x - 120$

Find the range of x such that

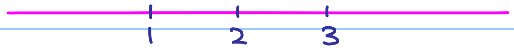
(1) $f'(x) > 0$, $f'(x) < 0$

(2) $f''(x) > 0$, $f''(x) < 0$

Step 1 : Find $f'(x)$ and factorize it.

$$f'(x) = 60x^4 - 420x^3 + 1020x^2 - 1020x + 360$$
$$= 60(x^4 - 7x^3 + 17x^2 - 17x + 6)$$
$$= 60(x-1)^2(x-2)(x-3)$$ (Using factor theorem)

Step 2:



gives intervals $x<1$, $1<x<2$, $2<x<3$, $x>3$

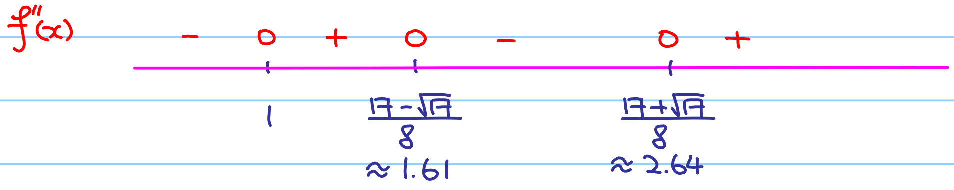
Reason : those factors may change sign at the boundary points of intervals.

Step 3:	$x<1$	$x=1$	$1<x<2$	$x=2$	$2<x<3$	$x=3$	$x>3$
$(x-1)^2$	+	0	+	+	+	+	+
$(x-2)$	−	−	−	0	+	+	+
$(x-3)$	−	−	−	−	−	0	+
$f'(x)$	+	0	+	0	−	0	+
$f(x)$	inc	saddle pt.	inc.	max.	dec.	min	inc.

saddle point = (1, −23) max = (2, −16) min = (3, −39)

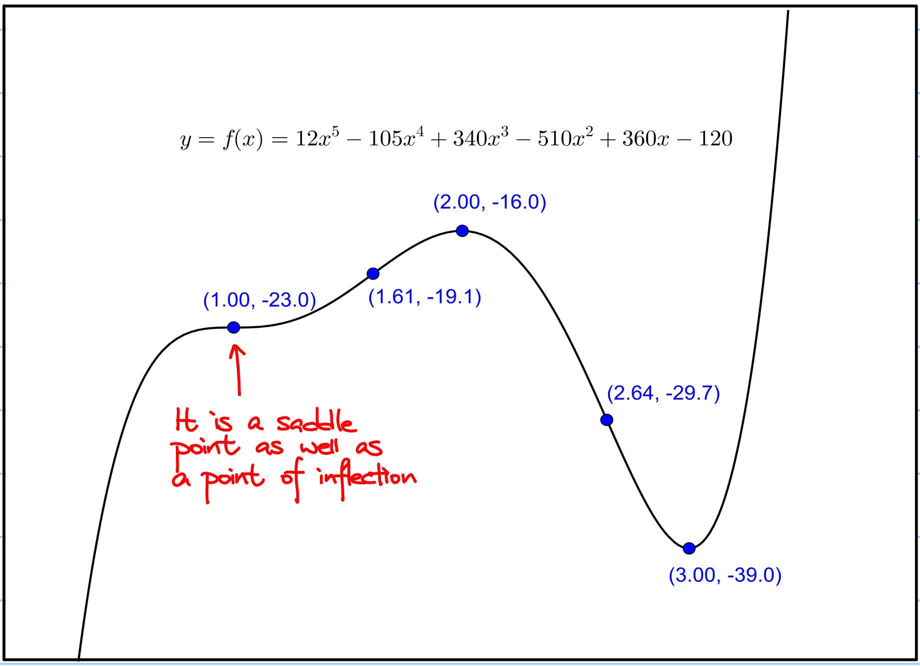
Similarly,

$$f''(x) = 240x^3 - 1260x^2 + 2040x - 1020$$
$$= 60(x-1)(4x^2 - 17x + 17)$$
$$= 240(x-1)\left[x - \left(\frac{17+\sqrt{17}}{8}\right)\right]\left[x - \left(\frac{17-\sqrt{17}}{8}\right)\right]$$



points of inflection : $(1,-23)$, $\left(\frac{17\pm\sqrt{17}}{8}, f\left(\frac{17\pm\sqrt{17}}{8}\right)\right)$

$= (1.61, -19.1)$ or $(2.64, -29.7)$



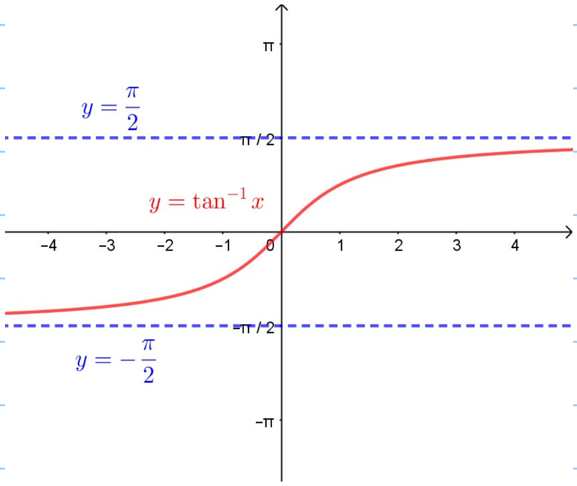
Exercise 6.5 1

Let $f(x) = \tan^{-1}x$. Show that

(a) $f'(x) > 0$ $\forall x \in \mathbb{R}$, so there is no stationary point ($\Rightarrow$ no saddle point)

(b) f has a point of inflection at $x=0$.

Therefore, saddle point and point of inflection are different concepts.



Example 6.5.3

$f(x) = \frac{x}{(x+1)^2}$, $x \neq -1$.

$f'(x) = \frac{1-x}{(x+1)^3}$

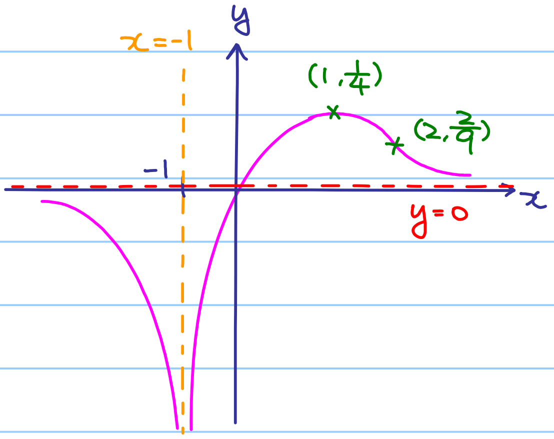
		-1		1	
$f'(x)$	$-$	NOT defined	$+$	0	$-$
$\Downarrow$					
$f(x)$	dec.	NOT defined	inc.	max.	dec.

max $= (1, \frac{1}{4})$

$f''(x) = \frac{2(x-2)}{(x+1)^4}$

		-1		2	
$f''(x)$	$-$	NOT defined	$-$		$+$
$\Downarrow$					
$f(x)$	∩		∩		∪

point of inflection : $(2, \frac{2}{9})$



Note : The graph of $y=f(x)$ behaves like :

- the vertical line $x=-1$, when x is "near" -1.
- the horizontal line $y=0$, when x is "near $+\infty$ or $-\infty$".

In fact , $x=-1$ is called a vertical asymptote,

$y=0$ is called a horizontal asymptote .

6.6 Asymptotes

Definition 6.6.1

1) If $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x) = +\infty$ or $-\infty$,

then $x=a$ is said to be a vertical asymptote.

2) If $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, where $L \in \mathbb{R}$,

then $y=L$ is said to be a horizontal asymptote.

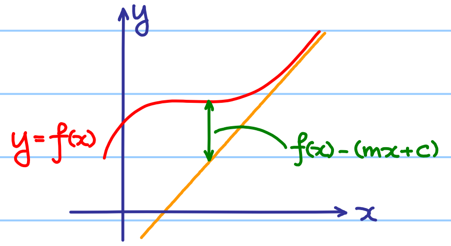
Note : It may happen that both $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist

but they are NOT the same.

3) If $y=mx+c$ is a straight such that $\lim_{x\to+\infty} f(x)-(mx+c)=0$ or $\lim_{x\to-\infty} f(x)-(mx+c)=0$,

then the straight line is said to be an oblique asymptote of $f(x)$.

Remark: In particular, if $m=0$, the asymptote is horizontal



the distance tends to 0

as $x \to +\infty$

Exercise 6.6.1

Prove that

(a) if $y=mx$ is an oblique asymptote of $f(x)$ at $+\infty$,

then $m = \lim_{x\to+\infty} \frac{f(x)}{x}$ and $c = \lim_{x\to+\infty} f(x)-mx$;

(b) if $\lim_{x\to+\infty} \frac{f(x)}{x}$ exists (denoted by m) and $\lim_{x\to+\infty} f(x)-mx$ exists (denoted by c),

then $y=mx$ is an oblique asymptote of $f(x)$ at $+\infty$.

Example 6.6.1

Let $f(x) = \frac{x|x-2|}{x-1}$, $x \neq 1$.

$$f(x) = \begin{cases} \frac{x(x-2)}{x-1} & \text{if } x \geq 2 \\ -\frac{x(x-2)}{x-1} & \text{if } x < 2 \text{ and } x \neq 1 \end{cases}$$

(a) Show that f is NOT differentiable at $x=2$.

Hint : Show that $\lim_{\Delta x\to 0} \frac{f(2+\Delta x)-f(2)}{\Delta x}$ does NOT exist.

(b) $f'(x) = \begin{cases} \dfrac{x^2-2x+2}{(x-1)^2} & \text{if } x > 2 \\ -\dfrac{x^2-2x+2}{(x-1)^2} & \text{if } x < 2 \text{ and } x \neq 1 \end{cases}$

Solve $f'(x) > 0$ and $f'(x) < 0$

Ans : $f'(x) > 0$ when $x > 2$

$f'(x) < 0$ when $x < 2$ and $x \neq 1$

min = $(2, 0)$

(c) $f''(x) = \begin{cases} \dfrac{-2}{(x-1)^3} & \text{if } x > 2 \\ \dfrac{2}{(x-1)^3} & \text{if } x < 2 \text{ and } x \neq 1 \end{cases}$

Solve $f''(x) > 0$ and $f''(x) < 0$

Ans : $f''(x) > 0$ when $1 < x < 2$

$f''(x) < 0$ when $x > 2$ or $x < 1$

point of inflection = $(2, 0)$

(d) vertical asymptote : $x = 1$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} -\frac{x(x-2)}{x-1} = -\infty$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} -\frac{x(x-2)}{x-1} = +\infty$$

oblique / horizontal asymptote :

① For $x \geq 2$, $f(x) = \dfrac{x(x-2)}{x-1}$

$$m = \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{x-2}{x-1} = 1$$

$$c = \lim_{x \to +\infty} f(x) - mx = \lim_{x \to +\infty} \frac{x(x-2)}{x-1} - x = \lim_{x \to +\infty} \frac{-x}{x-1} = -1$$

$\therefore$ $y = x - 1$ is an oblique asymptote.

② For $x < 2$ and $x \neq 1$, $f(x) = -\dfrac{x(x-2)}{x-1}$

$$m = \lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} -\frac{x-2}{x-1} = -1$$

$$c = \lim_{x \to -\infty} f(x) - mx = \lim_{x \to +\infty} -\frac{x(x-2)}{x-1} + x = \lim_{x \to +\infty} \frac{x}{x-1} = 1$$

$\therefore$ $y = -x + 1$ is an oblique asymptote.

(e) x-intercept : Solve $f(x) = 0$

$$\frac{x|x-2|}{x-1} = 0$$

$$x = 0 \text{ or } 2$$

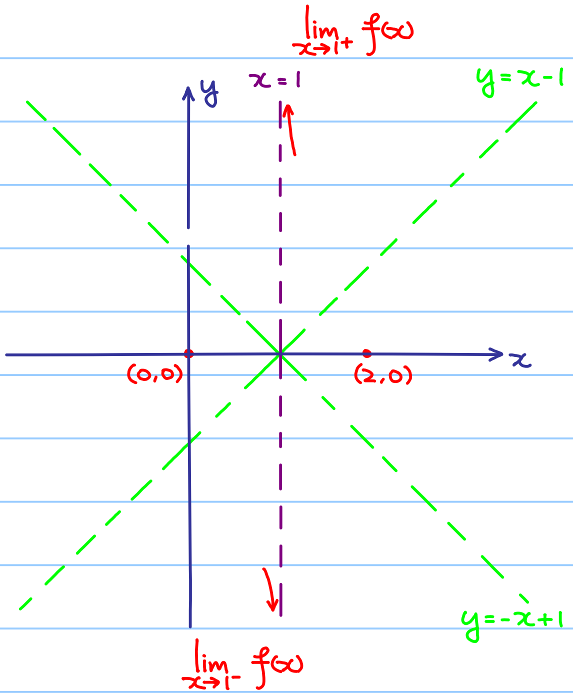
y-intercept : $f(0) = 0$.

(f) Sketch $y = f(x)$.

Step 1 : draw asymptotes

Step 2 : put down x-intercepts

and y-intercept

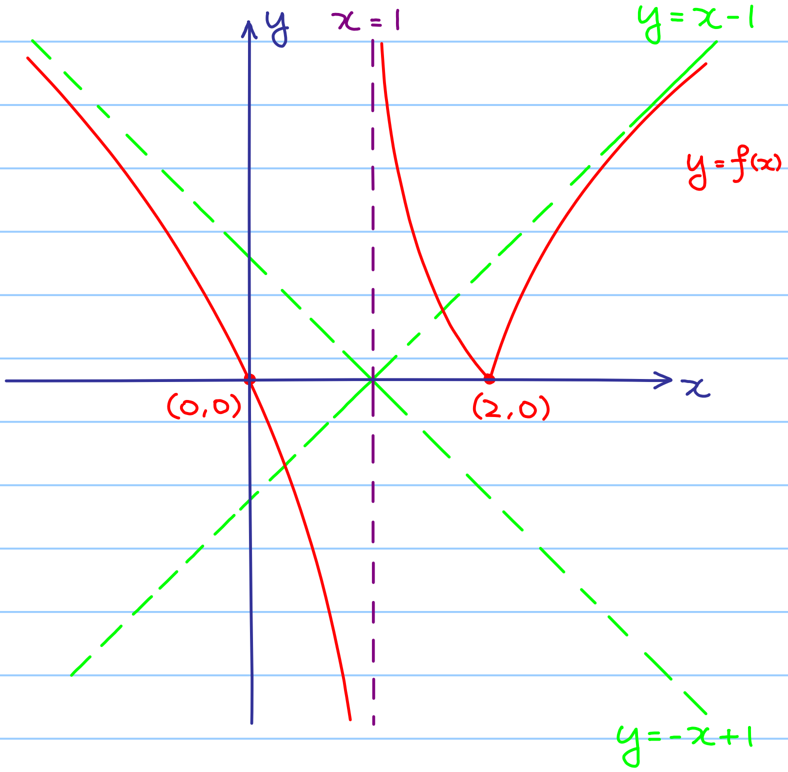


Step 3 :

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -\frac{x(x-2)}{x-1} = -\infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -\frac{x(x-2)}{x-1} = +\infty$$

Step 4 : Use the information $f'(x)$ and $f''(x)$



		1		2	
$f'(x)$	$-$	NOT defined	$-$	NOT defined	$+$
⇓					
$f(x)$	dec.		dec.		inc.

		1		2	
$f''(x)$	$-$	NOT defined	$+$	NOT defined	$-$
⇓					
$f(x)$	convex		concave		convex

Curve Sketching :

Goal : Given a function $f(x)$, sketch the graph of $y=f(x)$.

(Capturing main features)

• x - intercept	solve $f(x)=0$
• y - intercept	y-intercept $= f(0)$
• increasing / decreasing	solve $f'(x)>0$ / $f'(x)<0$
saddle point / max. / min	change of sign of $f'(x)$?
• concave / convex	solve $f''(x)>0$ / $f''(x)<0$
point of inflection	change of sign of $f''(x)$?
• vertical asymptote	any $x=a$ with $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$
• horizontal asymptote	$m = \lim_{x\to +\infty} \frac{f(x)}{x}$
• oblique asymptote	$c = \lim_{x\to +\infty} f(x) - mx$

§ 7 Taylor's Theorem and L'Hôpital's Rule

7.1 Talyor Polynomials

Let $f(x)$ be a function with derivatives of all orders on an open interval I, and $a \in I$.

Goal: Can we approximate $f(x)$ around the point $x = a$ by a polynomial $P_n(x)$ of degree n

in a sense that

$$\left.\begin{aligned} f(a) &= P_n(a) \\ f'(a) &= P_n'(a) \\ &\vdots \\ f^{(n)}(a) &= P_n^{(n)}(a) \end{aligned}\right\} \text{n+1 conditions} \quad ?$$

i.e. $f(x)$ and $P_n(x)$ agree with each other up to n-th derivative at $x = a$.

Let $P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$

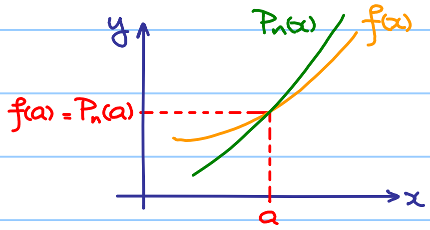
$$= \sum_{k=0}^{n} c_k (x-a)^k$$

$\underbrace{c_0, c_1, \cdots, c_n}_{\text{n+1 constants}}$ are constants to be detemined.

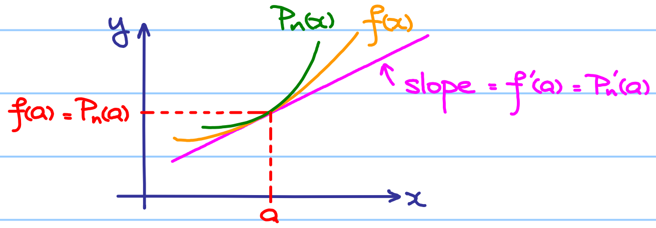
n+1 constants.

Remark: n+1 conditions, n+1 constants $\Rightarrow$ c_k's are completely determined.

Idea: If $f(a) = P_n(a)$, then the graphs of $f(x)$ and $P_n(x)$ intersect at $x = c$.



Furthermore, if $f(a) = P_n(a)$, then the graphs of f and P_n share the same tangent line at $x = a$.



$f(x)$ and $P_n(x)$ agree with each other up to n-th derivative at $x = a$ is the generalization of the above, hopefully increasing n (the degree of $P_n(x)$) will give better approximation of $f(x)$ around $x = a$.

To determine a_i's :

- $P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$

  $f(c) = P_n(c) = c_0$

- $P_n'(x) = 1\,c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1}$

  $f'(a) = P_n'(a) = 1\,c_1$

  $c_1 = \frac{f'(a)}{1!}$

- $P_n''(x) = 2\cdot 1\,c_2 + 3\;2\,c_3(x-a) + \cdots + n\cdot(n-1)c_n(x-a)^{n-2}$

  $f''(a) = P_n''(a) = 2!\,c_2$

  $c_2 = \frac{f''(a)}{2!}$

Repeating the process, in general, we have $c_k = \frac{f^{(k)}(a)}{k!}$ $\quad k = 0, 1, 2, \cdots, n$

Definition 7.1.1

Let I be an open interval and $a \in I$.

Let $f: I \to \mathbb{R}$ be a function such that $f'(a), f''(a), \cdots, f^{(n)}(a)$ exist.

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the Talyor polynomial of order n generated by f at $x = a$

In particular, $y = P_1(x) = f(a) + f'(a)(x-a)$ is the tangent line of $f(x)$ at $x = 0$.

Example 7.1.1

Let $f(x) = e^x$, find the Talyor polynomials $P_n(x)$ generated by f at $x=0$.

Note : $f^{(k)}(x) = e^x$ and $f^{(k)}(0) = 1$ for $k = 0, 1, 2, \cdots, n$

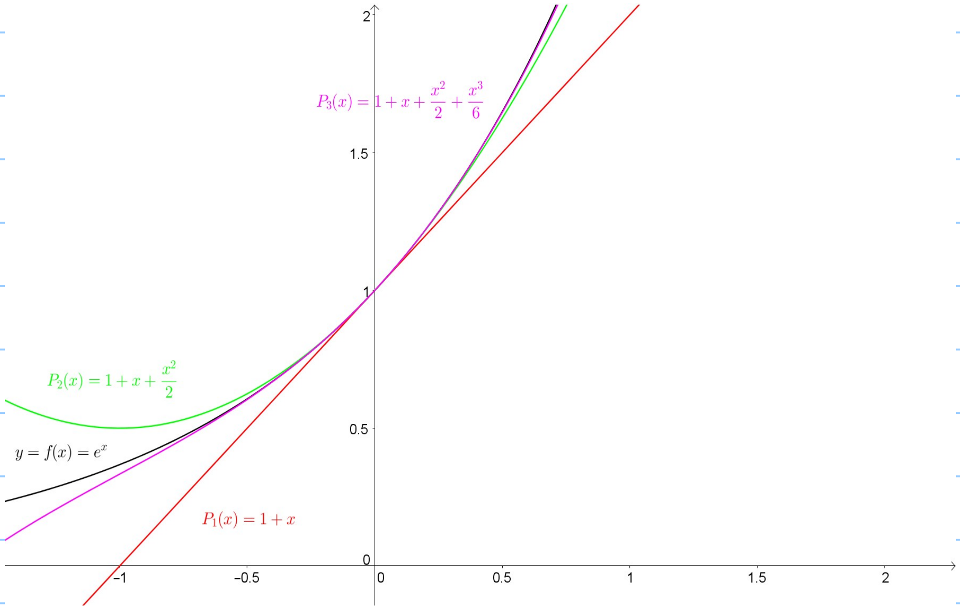
$$P_0(x) = f(0) \qquad = 1$$

$$P_1(x) = f(0) + \frac{f'(0)}{1!}x \qquad = 1 + x$$

$$P_2(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 \qquad = 1 + x + \frac{x^2}{2}$$

$$P_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

In particular, $y = P_1(x) = 1 + x$ is the tangent line of $f(x)$ at $x = 0$.



In general, $P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$

$$= 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n$$

$$= \sum_{k=0}^{n} \frac{1}{k!}x^k$$

Example 7.1.2

Let $f(x) = \cos x$, find the Talyor polynomials $P_n(x)$ generated by f at $x=0$.

Note : $f(x) = \cos x$ $\qquad f'(x) = -\sin x$

$f''(x) = -\cos x$ $\qquad f'''(x) = \sin x$

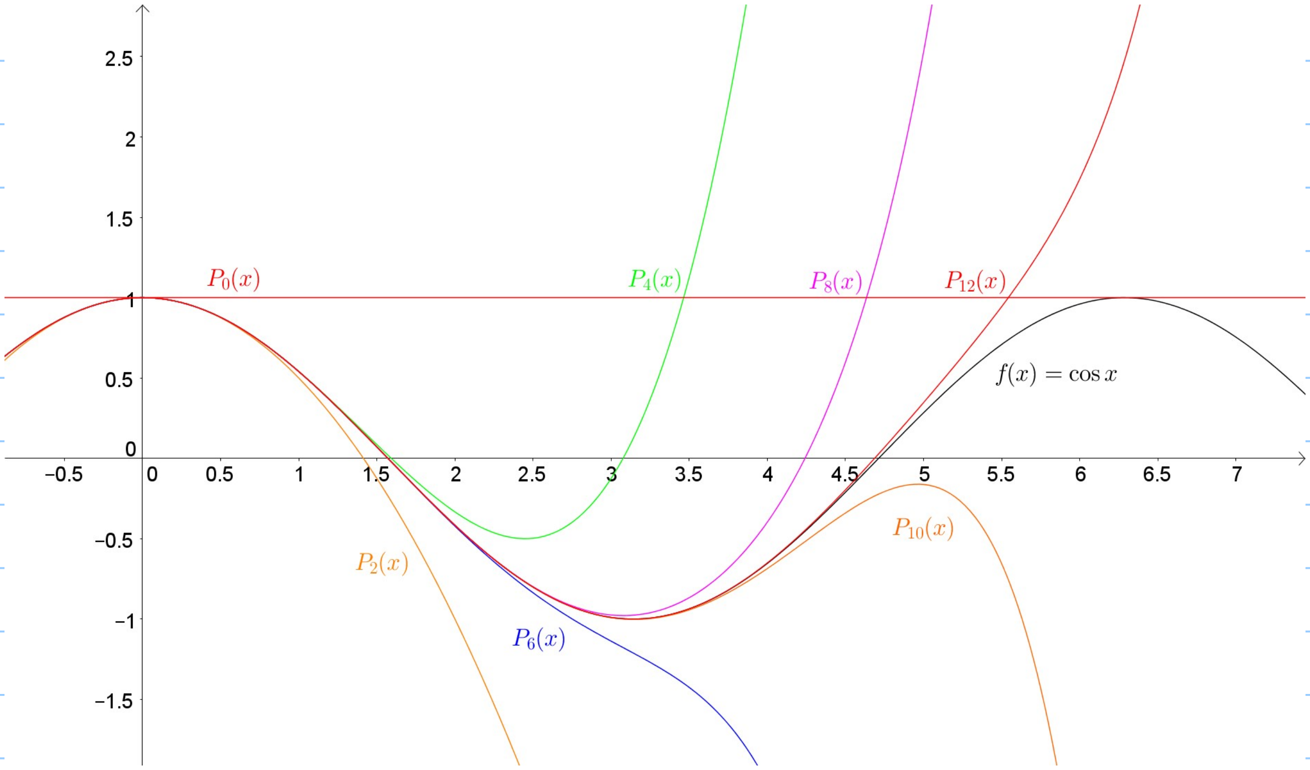
$f^{(4)}(x) = \cos x$ $\qquad f^{(5)}(x) = -\sin x$

$\vdots \qquad \vdots$

$f^{(2n)}(x) = (-1)^n \cos x$ $\qquad f^{(2n+1)}(x) = (-1)^{n+1} \sin x$

$\therefore$ $f^{(2n)}(0) = (-1)^n$ $\qquad f^{(2n+1)}(0) = 0$

$$\therefore P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$



Exercise 7.1.1

Find the Talyor polynomials $P_n(x)$ generated by f at $x=0$ if $f(x) =$

a) $\sin x$

b) $\frac{1}{1-x}$

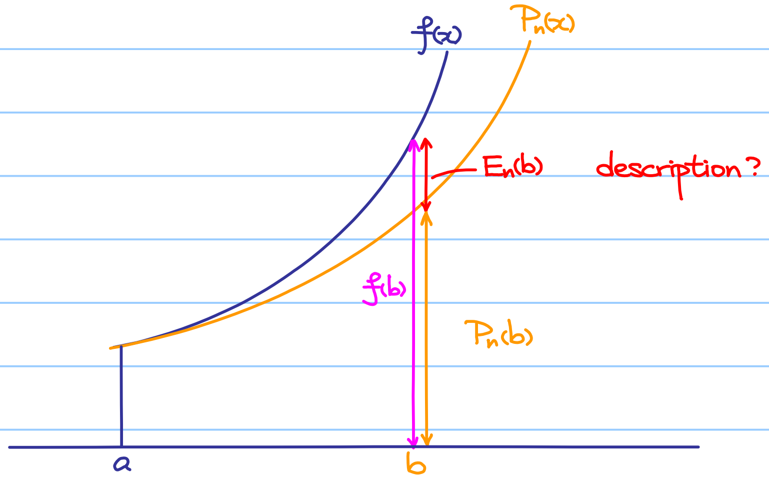
c) $\ln(1+x)$

Idea :

We want to approximate $f(b)$ by $P_n(b)$, there is an error term $E_n(b) = f(b) - P_n(b)$

The error term tells us how good / bad our approximation is !

Therefore, we need a description of the error term $E_n(b)$.



7.2 Talyor's Theorem

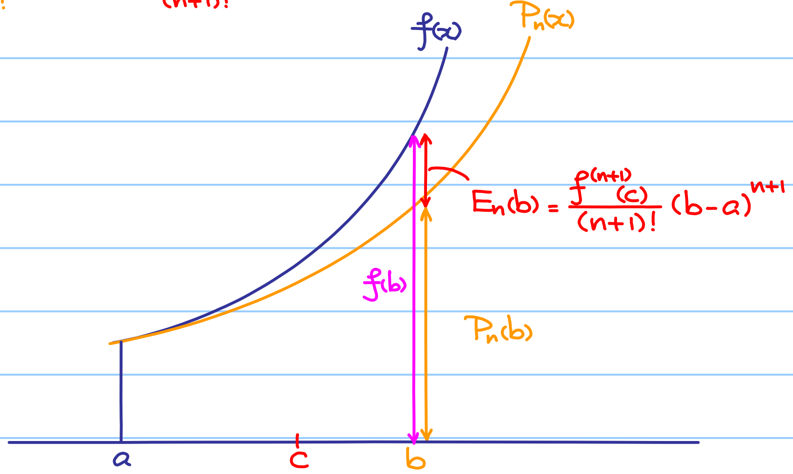
Theorem 7.2.1 (Talyor's Theorem)

If f and its first n derivatives $f', f'', \dots, f^{(n)}$ are continuous on the closed interval between a and b, $f^{(n)}$ is differentiable on the open interval between a and b, then there exists c between a and b such that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

$$= P_n(b) + E_n(b)$$

i.e. the error can be described by the (n+1)-th derivative of f.



Remark: If $n = 0$,

$$f(b) = f(a) + f'(c)(b-a)$$

$$= P_0(b) + E_0(b)$$

It is just Mean Value Theorem !

Therefore, Taylor's theorem can be regarded as a generalization of the mean value theorem.

proof:

Assume $b > a$.

Let $F(x) = f(x) - P_n(x) - \frac{f(b)-P_n(b)}{(b-a)^{n+1}}(x-a)^{n+1}$

Check: F is continuous on $[a,b]$

F is differentiable on (a,b)

$F(a) = F(b) = 0$

Apply Rolle's Theorem, $\exists\, c_1 \in (a,b)$ such that $F'(c_1) = 0$

Check: F' is continuous on $[a,b]$

F' is differentiable on (a,b)

$F'(a) = F'(c_1) = 0$

Apply Rolle's Theorem, $\exists\, c_2 \in (a,c_1)$ such that $F''(c_2) = 0$

Repeating the process : $\exists\, c_{n+1} \in (a, c_n)$ such that $F^{(n+1)}(c_{n+1}) = 0$

Note: $F^{(n+1)}(x) = f^{(n+1)}(x) - (n+1)!\,\frac{f(b)-P_n(b)}{(b-a)^{n+1}}$

$$0 = F^{(n+1)}(c_{n+1}) = f^{(n+1)}(c_{n+1}) - (n+1)!\,\frac{f(b)-P_n(b)}{(b-a)^{n+1}}$$

$$\frac{f^{(n+1)}(c_{n+1})}{(n+1)!} = \frac{f(b)-P_n(b)}{(b-a)^{n+1}}$$

$$\therefore\; f(b) = P_n(b) + \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}(b-a)^{n+1}$$

The proof for the case $a > b$ is similar.

Example 7.2.1

Approximate cos 0.1

Let $f(x) = \cos x$,

$P_5(x)\ (=P_4(x)) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ Talyor polynomials of degree 5 generated by f at $x=0$.

$\cos 0.1 = f(0.1) \approx P_5(0.1) = 0.995004166\cdots$

By Talyor's Theorem $f(0.1) = P_5(0.1) + \frac{f^{(6)}(c)}{6!}(0.1)^6$ $c \in (0, 0.1)$

Absolute Error $= |E_5(0.1)| = \left|\frac{f^{(6)}(c)}{6!}(0.1)^6\right|$

$\le \frac{1}{6!}(0.1)^6 \approx 1.38 \times 10^{-9}$

Very small.

Note : $f^{(6)}(x) = -\cos x$

$\Rightarrow |f^{(6)}(c)| \le 1$

get rid of c !

Example 7.2.2

Let $f(x) = \frac{1}{1-x}$

Suppose $P_n(x)$ is Talyor polynomial of order n generated by f at 0.

Then $P_n(x) = 1 + x + x^2 + \cdots + x^n$

Note : $f(0.1) = \frac{1}{1-0.1} = 1.111\cdots$

$P_n(0.1) = 1 + 0.1 + 0.1^2 + \cdots + 0.1^n = 1.\underbrace{111\cdots1}_{n}$

$E_n(0.1) = f(0.1) - P_n(0.1)$ is getting closer and closer to 0 as n increases.

Good Approximation

$f(2) = \frac{1}{1-2} = -1$

$P_n(2) = 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

$E_n(2) = f(2) - P_n(2)$ is NOT getting closer and closer to 0 as n increases.

Bad Approximation

7.3 Talyor's Series

Idea :

For any fixed b, $E_n(b)$ becomes a sequence of real numbers.

If $\lim_{n\to\infty} E_n(b) = 0$, that means the error is getting closer and closer to 0

as we increase the number of terms to approximate $f(x)$.

However, it is not always true as we can see in example 7.2.2 .

Definition 7.3.1

Let I be an open interval, $a \in I$ and let $f: I \to \mathbb{R}$ be a function.

Suppose that $f^{(n)}(a)$ exists for all $n \in \mathbb{Z}^+$, then we define the Taylor series

generated by f at $x=a$ to be

$$T(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= \lim_{n\to\infty} P_n(x)$$

Example 7.3.1

Let $f(x) = \frac{1}{1-x}$

Then the Talyor series generated by f at 0 is

$$T(x) = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k$$

Question : What is the relation between $f(x)$ and $T(x)$?

Theorem 7.3.1

Let I be an interval, let a be an interior point of I, and let $f: I \to \mathbb{R}$ be a function.

Suppose that $f^{(n)}(a)$ exists for all $n \in \mathbb{Z}^+$,

$P_n(x)$ is the Taylor polynomial of order n generated by f at $x=a$ and $E_n(x) = f(x) - P_n(x)$.

If $\lim_{n\to\infty} E_n(x) = 0$ for all $x \in I$, then we have

$$\lim_{n\to\infty} f(x) - P_n(x) = 0$$

$$f(x) = \lim_{n\to\infty} P_n(x) = T(x)$$

ie. $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ for all $x \in I$

In this case, we say that the Talyor series converges to $f(x)$ for all $x \in I$

Example 7.3.2

Let $f(x) = \cos x$

$$f(x) = \cos x = \underset{\substack{\| \\ P_{2n}(x) = P_{2n+1}(x)}}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}} + \underset{\substack{\| \\ E_{2n}(x)}}{E_{2n+1}(x)}$$

$$0 \leq |E_{2n+1}(x)| = \left| \frac{f^{(2n+2)}(c(x))}{(2n+2)!} x^{2n+2} \right| \leq \frac{|x^{2n+2}|}{(2n+2)!}$$

Note · $|f^{(2n+2)}(c(x))| = |\cos(c(x))| \leq 1$

By theorem 2 4.4, $\lim_{n\to\infty} \frac{|x^{2n+2}|}{(2n+2)!} = 0$

By sandwich theorem, $\lim_{n\to\infty} |E_{2n+1}(x)| = 0$ and hence $\lim_{n\to\infty} E_{2n+1}(x) = 0$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k)!} x^{2k}$$ for all $x \in \mathbb{R}$

Frequently used Talyor series :

1) $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$, $|x| < 1$

2) $\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-x)^n$, $|x| < 1$

3) $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\forall x \in \mathbb{R}$

4) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $\forall x \in \mathbb{R}$

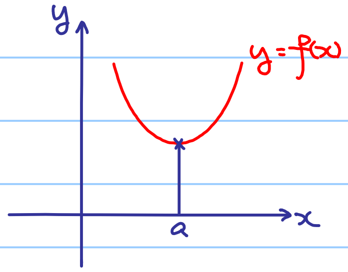
5) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, $\forall x \in \mathbb{R}$

6) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$, $\forall -1 < x \leq 1$

Example 7.3.3 (NOT Rigorous)

Suppose $f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$

in an interval I and a lies in the interior of I.



If we know $f'(a)=0$ and $f''(a)>0$,

then if $x\sim a$,

$$f(x)\approx f(a)+\overset{0}{\cancel{f'(a)}}(x-a)+\underbrace{\frac{f''(a)}{2!}}_{\underset{0}{\vee}}(x-a)^2$$

locally, like a parabola opening upward!

It suggests why $f(x)$ attains minimum at $x=a$.

How about $f''(a)=0$?

7.4 Manipulation of Taylor Series

Without caring the convergence, we have

Example 7.4.1

1) (Addition)

$$\cos x+\sin x=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)+\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)=1+x-\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots$$

2) (Subtraction)

$$\cos x-\sin x=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)=1-x-\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

3) (Product)

$$\begin{aligned}\cos x\sin x&=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)\\&=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)x+\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)\left(-\frac{x^3}{3!}\right)+\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)\left(\frac{x^5}{5!}\right)+\cdots\\&=x-\frac{x^3}{2!}+\frac{x^5}{4!}-\cdots\\&\qquad-\frac{x^3}{3!}+\frac{x^5}{2!3!}-\cdots\\&\qquad\qquad+\frac{x^5}{5!}-\cdots\\&=x-\frac{2}{3}x^3+\frac{2}{15}x^5-\cdots\end{aligned}$$

4) (Composition)

$$e^{\sin x} = 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^3 + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \cdots$$

5) (Differentiation)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\frac{d}{dx}\sin x = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

6) (Integration)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\int \sin x \, dx = \int x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots dx$$

$$-\cos x = \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots\right) + C$$

putting $x = 0$,

$$-1 = C$$

$$\therefore \; -\cos x = -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots$$

7) (Division)

Let $\dfrac{\sin x}{\cos x} = a_0 + a_1 x + a_2 x^2 + \cdots$

$$\therefore \; \sin x = \cos x \, (a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$\begin{array}{llllll}
= a_0 & & -\frac{a_0}{2!}x^2 & & +\frac{a_0}{4!}x^4 & - \cdots \\
& a_1 x & & -\frac{a_1}{2!}x^3 & & +\frac{a_1}{4!}x^5 \quad - \cdots \\
& & a_2 x^2 & & -\frac{a_2}{2!}x^4 & \quad +\frac{a_2}{4!}x^6 \quad - \cdots \\
& & & a_3 x^3 & & -\frac{a_3}{2!}x^5 \quad +\frac{a_3}{4!}x^7 \quad - \cdots
\end{array}$$

Compare coefficients of x^r for $r = 0, 1, 2, 3, \cdots$:

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \\ -\frac{a_0}{2!} + a_2 = 0 \\ -\frac{a_1}{2!} + a_3 = -\frac{1}{3!} \\ \quad \vdots \end{cases}$$

$$\therefore \; a_0 = 0, \; a_1 = 1, \; a_2 = 0, \; a_3 = \frac{1}{3}, \; \cdots$$

$$\tan x = \frac{\sin x}{\cos x} = x + \frac{1}{3}x^3 + \cdots$$

7.5 Indeterminate Form $\frac{0}{0}$ and L'hôpital's Rule

Example 7.5.1 (NOT Rigorous)

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x}$$

$$= 1 \underbrace{- \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots}_{\text{terms involves } x}$$

It suggests $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (How do we know $\lim_{x\to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots = 0$? There are infinitely many terms !)

In general, $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots$

Suppose $f(a) = g(a) = 0$ and $f'(a), g'(a) \neq 0$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots}{g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots}$$

$$= \lim_{x\to a} \frac{f'(a) + \text{terms involves } (x-a)}{g'(a) + \text{terms involves } (x-a)} = \frac{f'(a)}{g'(a)}$$

The formal statement : L'hôpital's Rule

Consider $\lim_{x\to a} \frac{f(x)}{g(x)}$ and suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

Case 1 : If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

Case 2 : If $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} f(x) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ does NOT exist. (e.g. $\lim_{x\to 1} \frac{x}{x-1}$)

Case 3 : If $\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = 0$, then we do NOT know whether $\lim_{x\to a} \frac{f(x)}{g(x)}$ exist !

e.g. $\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$,

$\lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$,

$\lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x}$ does not exist.

We call it indeterminate form $\frac{0}{0}$.

Theorem 7.5.1 (L'hôpital's Rule)

Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, I is an open interval containing a, f and g are differentiable on $I\setminus\{a\}$, and $g'(x) \neq 0$ on $I\setminus\{a\}$.

If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

(Furthermore, if $f'(x)$ and $g'(x)$ are continuous at a and $g'(a) \neq 0$,

then $\lim_{x\to a} f'(x) = f'(a)$ and $\lim_{x\to a} g'(x) = g'(a)$

$\therefore \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$)

Example 7.5.2

$\lim_{x\to 0} \frac{\sin x}{x}$ $(\frac{0}{0})$ — (*)

$= \lim_{x\to 0} \frac{\cos x}{1}$ — (**) Logic: limit (**) exists ⇒ limit (*) exists

$= \frac{1}{1}$

$= 1$

Example 7.5.3

$\lim_{x\to 0} \frac{1-\cos x}{x^2}$ $(\frac{0}{0})$

$= \lim_{x\to 0} \frac{\sin x}{2x}$ $(\frac{0}{0})$

$= \lim_{x\to 0} \frac{\cos x}{2}$

$= \frac{1}{2}$

Example 7.5.4 (Non-example)

Clearly, $\lim_{x\to 1} \frac{2x+3}{x-1}$ does NOT exist.

However, some may misuse L'Hopital's rule:

$\lim_{x\to 1} \frac{2x+3}{x-1}$

$= \lim_{x\to 1} \frac{2}{1}$ ✗ Since the above limit is NOT in the indeterminate form $\frac{0}{0}$

$= 2$

Example 7.5.4 (L'Hopital's rule fails)

$$\lim_{x\to0}\frac{x^2\sin(\frac{1}{x})}{\sin x} \quad (\tfrac{0}{0})$$

$$=\lim_{x\to0}\frac{2x\sin(\frac{1}{x})-\cos(\frac{1}{x})}{\cos x}$$ and this limit does not exist.

However, $$\lim_{x\to0}\frac{x^2\sin(\frac{1}{x})}{\sin x}=\lim_{x\to0}\frac{x\sin(\frac{1}{x})}{(\frac{\sin x}{x})}$$

$$=\frac{\lim_{x\to0}x\sin(\frac{1}{x})}{\lim_{x\to0}\frac{\sin x}{x}}$$

$$=\frac{0}{1}=0$$

7.6 Indeterminate Form $\frac{\infty}{\infty}$, $\infty\,0$, $\infty-\infty$

- L'hôpital's Rule can also be applied to $\frac{\infty}{\infty}$
- L'hôpital's Rule can also be applied to left hand limit or right hand limit

$$\lim_{x\to a^+}\frac{f(x)}{g(x)}=\lim_{x\to a^+}\frac{f'(x)}{g'(x)}, \qquad \lim_{x\to a^-}\frac{f(x)}{g(x)}=\lim_{x\to a^-}\frac{f'(x)}{g'(x)}$$

- L'hôpital's Rule can also be applied to limits at infinities

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}, \qquad \lim_{x\to-\infty}\frac{f(x)}{g(x)}=\lim_{x\to-\infty}\frac{f'(x)}{g'(x)}$$

Example 7.6.1

$$\lim_{x\to\frac{\pi}{2}^-}\frac{\sec x}{1+\tan x} \quad (\tfrac{\infty}{\infty})$$

$$=\lim_{x\to\frac{\pi}{2}^-}\frac{\sec x\tan x}{\sec^2 x}$$

$$=\lim_{x\to\frac{\pi}{2}^-}\sin x$$

$$=1$$

Example 7.6.2

$$\lim_{x\to+\infty}\frac{\ln x}{2\sqrt{x}} \quad (\tfrac{\infty}{\infty})$$

$$=\lim_{x\to+\infty}\frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}}$$

$$=\lim_{x\to+\infty}\frac{1}{\sqrt{x}}$$

$$=0$$

Indeterminate Form $\infty\ 0$

Idea : Converting to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example 7.6.3

$\lim\limits_{x\to+\infty} x\sin\frac{1}{x}$ $\quad(\infty\ 0)$

↓ convert to

$= \lim\limits_{x\to+\infty} \dfrac{\sin\frac{1}{x}}{\frac{1}{x}}$ $\quad(\frac{0}{0})$

$= \lim\limits_{x\to+\infty} \dfrac{-\frac{1}{x^2}\cos\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim\limits_{x\to+\infty} \cos\frac{1}{x}$

$= 1$

Alternative method :

$\lim\limits_{x\to+\infty} x\sin\frac{1}{x}$ $\quad(\infty\ 0)$

Let $h=\frac{1}{x}$,

As $x\to+\infty$, $h\to0^+$

$= \lim\limits_{h\to0^+} \dfrac{\sin h}{h}$ $\quad(\frac{0}{0})$

$= 1$

Example 7.6.4

$\lim\limits_{x\to0^+} \sqrt{x}\ln x$ $\quad(\infty\ 0)$

↓ convert to

$= \lim\limits_{x\to0^+} \dfrac{\ln x}{\frac{1}{\sqrt{x}}}$ $\quad(\frac{\infty}{\infty})$

$= \lim\limits_{x\to0^+} \dfrac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}}$

$= \lim\limits_{x\to0^+} -2\sqrt{x}$

$= 0$

Remark: Why don't we try $\lim\limits_{x\to0^+}\sqrt{x}\ln x = \lim\limits_{x\to0^+}\dfrac{\sqrt{x}}{(\frac{1}{\ln x})}$ $\quad(\frac{0}{0})$?

Indeterminate Form $\infty - \infty$

Idea : Converting to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example 7.6.5

$\lim\limits_{x\to0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$ $\quad(\infty-\infty)$

) convert to

$= \lim\limits_{x\to0} \dfrac{x-\sin x}{x\sin x}$ $\quad(\frac{0}{0})$

Ex: $\vdots$

$= 0$

7.7 Indeterminate Form 1^{∞}, 0^{0}, ∞^{0}

Indeterminate Form 1^{∞}, 0^{0}, ∞^{0}

Idea: Taking ln, converting to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example 7.7.1

Find $\lim_{x\to 1^+} x^{\frac{1}{1-x}}$ $\quad (1^{\infty})$

Let $y = x^{\frac{1}{1-x}}$

$\ln y = \frac{\ln x}{1-x}$

$\lim_{x\to 1^+} \ln y = \lim_{x\to 1^+} \frac{\ln x}{1-x}$ $\quad (\frac{0}{0})$

$\ln(\lim_{x\to 1^+} y) = \lim_{x\to 1^+} \frac{(\frac{1}{x})}{-1}$

$= -1$

$\ln(\lim_{x\to 1^+} x^{\frac{1}{1-x}}) = -1$

$\therefore \lim_{x\to 1^+} x^{\frac{1}{1-x}} = e^{-1}$

Example 7.7.2

Find $\lim_{x\to +\infty} x^{\frac{1}{x}}$ $\quad (\infty^{0})$

Let $y = x^{\frac{1}{x}}$

$\ln y = \frac{\ln x}{x}$

$\lim_{x\to +\infty} \ln y = \lim_{x\to +\infty} \frac{\ln x}{x}$ $\quad (\frac{\infty}{\infty})$

$\ln(\lim_{x\to +\infty} y) = \lim_{x\to +\infty} \frac{(\frac{1}{x})}{1}$

$\ln(\lim_{x\to +\infty} x^{\frac{1}{x}}) = 0$

$\therefore \lim_{x\to +\infty} x^{\frac{1}{x}} = e^{0} = 1$

Exercise 7.7.1

Show that $\lim_{x\to +\infty} (1+\frac{1}{x})^{x} = e$ by L'Hoprtal's rule.

§ 8 Indefinite Integration

8.1 Antiderivatives

Definition 8.1.1

A function $F(x)$ is said to be an antiderivative of $f(x)$ if $F'(x) = f(x)$.

The process of finding antiderivatives is called indefinite integration.

Example 8.1.1

If $f(x) = 2x$, $F(x) = x^2$,

then we have $F'(x) = f(x)$, so $F(x)$ is an antiderivative of $f(x)$.

However, consider $F(x) = x^2 + C$, where C is a constant.

Then, we still have $F'(x) = f(x)$.

Therefore, antiderivative of a function $f(x)$ is NOT unique.

That is why we call "an" antiderivative instead of "the" antiderivative.

Natural question : If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$,
what is the relation between them ?

Answer : $F(x)$ and $G(x)$ differ by a constant.

proof : Suppose $F'(x) = G'(x) = f(x)$

Let $H(x) = F(x) - G(x)$

Then $H'(x) = F'(x) - G'(x) = 0$

$\therefore$ $H(x)$ is a constant function, i.e. $H(x) = C$ for some constant C.

i.e $F(x) = G(x) + C$

(Refer to theorem 6.2.2)

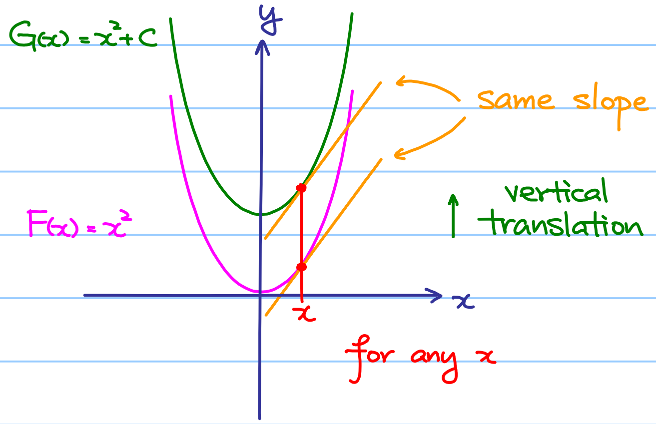
Therefore, antiderivative of a function $f(x)$ is NOT unique,
but it is unique up to a constant.

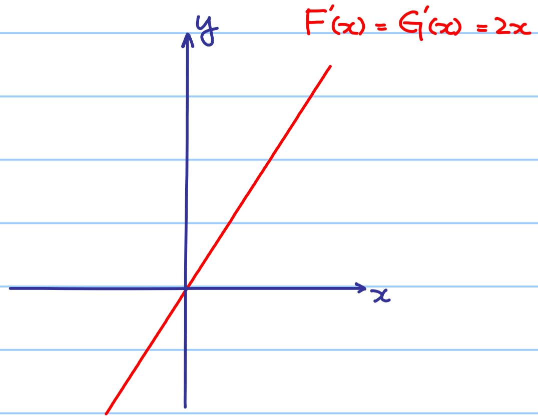
Example 8.1.2

If $f(x) = 2x$, $F(x) = x^2$

then we have $F'(x) = f(x)$, so $F(x) = x^2$ is an antiderivative of $f(x) = 2x$

and all antiderivatives of $f(x)$ must be of the form $x^2 + C$.





If F(x) is an antiderivative of f(x) , we write

$$\overset{\text{integrand}}{\downarrow}$$

$$\int f(x)\,dx = F(x) + C$$

integral symbol variable of integration

Example 8.1.2

$$\int 2x\,dx = x^2 + C$$

Note : When we write $\int f(x)\,dx = F(x) + C$, it should be regarded as a class of functions.

Furthermore, $\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}(F(x) + C) = f(x)$ no matter what C is.

8.2 Rules of Indefinite Integration

Theorem 8.2.1

1) $\int k\,dx = kx + C$, for a constant k.

2) $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, for all $n \neq -1$.

3) $\int \frac{1}{x}\,dx = \ln|x| + C$

4) $\int e^x\,dx = e^x + C$

5) $\int \cos x\,dx = \sin x + C$

6) $\int \sin x\,dx = -\cos x + C$

7) $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$

proof:

Derivative of RHS = Integrand on LHS

Theorem 8.2.2

1) $\int kf(x)\,dx = k\int f(x)\,dx$

2) $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Remark: We are proving two classes of functions are the same.

proof:

1) $\frac{d}{dx}(\text{LHS}) = \frac{d}{dx}\left(\int kf(x)\,dx\right) = kf(x)$

$\frac{d}{dx}(\text{RHS}) = \frac{d}{dx}\left(k\int f(x)\,dx\right) = k\frac{d}{dx}\int f(x)\,dx = kf(x)$

$\therefore$ (By theorem 6 2.2) LHS = RHS + C

$\int kf(x)\,dx = k\int f(x)\,dx$ (The constant C is dropped since we are comparing two classes of functions.)

2) Similarly, the result follows by $\frac{d}{dx}\left(\int f(x) \pm g(x)\,dx\right) = \frac{d}{dx}\left(\int f(x)\,dx \pm \int g(x)\,dx\right) = f(x) \pm g(x)$.

Example 8.2.1

$\int 2x^5 - 3x^2 + 7x + 5\,dx$

$\int dx$ means $\int 1\,dx$

$= 2\int x^5\,dx - 3\int x^2\,dx + 7\int x\,dx + 5\int dx$

$\int$ still there,

No need to add $+C$!

$= 2\left(\frac{x^6}{6}\right) - 3\left(\frac{x^3}{3}\right) + 7\left(\frac{x^2}{2}\right) + 5x + C$

$= \frac{x^6}{3} - x^3 + \frac{7x^2}{2} - 5x + C$

Example 8.2.2

$$\int \frac{x^3-5}{x}\,dx$$

$$= \int x^2 - \frac{5}{x}\,dx$$

$$= \frac{x^3}{3} - 5\ln|x| + C$$

Example 8.2.3

Find a function $F(x)$ such that $F(0)=3$ and $F'(x)=2x$.

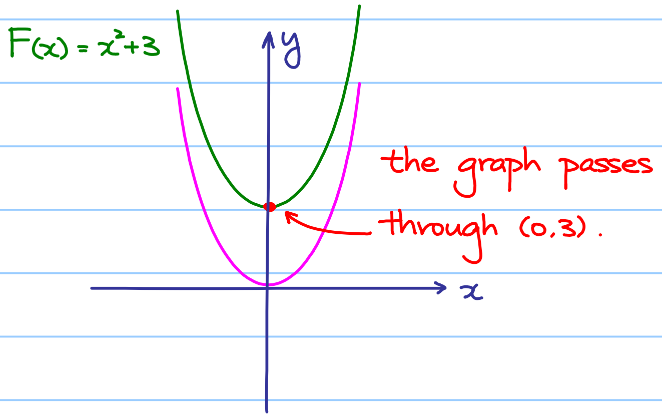
$F'(x) = 2x$

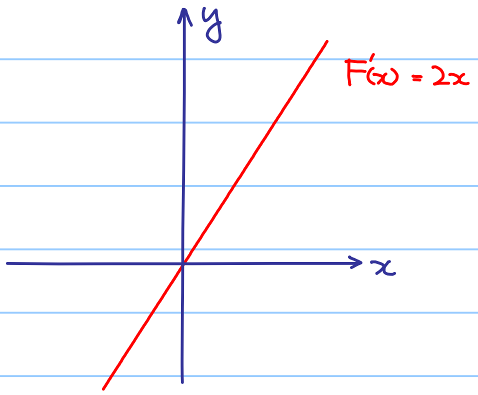
$F(x) = \int 2x\,dx$

$\quad = x^2 + C$

$F(0) = 0^2 + C = 3 \Rightarrow C = 3$

$\therefore F(x) = x^2+3$





8.3 Integration by Substitution

Question : $\int (2x+1)^{10}\,dx = ?$

Hard to integrate by expanding the polynomial.

Solution : Integration by Substitution

Theorem 8.3.1

$\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$ OR : $\int f(x)\frac{du}{dx}\,dx = \int f(u)\,du$

proof :

$$\frac{d}{dx}\int f(u(x))\,u'(x)\,dx = f(u(x))\,u'(x)$$

$$\frac{d}{dx}\int f(u)\,du = \frac{d}{du}\int f(u)\,du\ \frac{du}{dx} \qquad \text{(Chain Rule)}$$

$$= f(u(x))\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\int f(u(x))\,u'(x)\,dx = \frac{d}{dx}\int f(u)\,du$$

$$\therefore \int f(u(x))\,u'(x)\,dx = \int f(u)\,du$$

Example 8.3.1

$\int (2x+1)^{10}\,dx = ?$

Let $u(x) = 2x+1$ $\quad u'(x) = 2$

$f(u) = u^{10}$ $\quad f(u(x)) = (2x+1)^{10}$

$$\int (2x+1)^{10}\,dx = \frac{1}{2}\int \underset{\substack{\uparrow\\ f(u(x))}}{(2x+1)^{10}}\cdot\underset{\substack{\uparrow\\ u'(x)}}{2}\,dx = \frac{1}{2}\int \underset{\substack{\uparrow\\ f(u)}}{u^{10}}\,du$$

$$= \frac{1}{22}u^{11} + C = \frac{1}{22}(2x+1)^{11} + C$$

But, usually we write,

$\int (2x+1)^{10}\,dx$

$= \int u^{10}\cdot\frac{1}{2}du$

$= \frac{1}{22}u^{11} + C$

$= \frac{1}{22}(2x+1)^{11} + C$

Let $u = 2x+1$

$\frac{du}{dx} = 2$

$dx = \frac{1}{2}du$

(called differential form, can be defined rigorously)

Example 8.3.2

$\int e^{ax}\,dx$ Let $u = ax$

$= \int e^{u} \cdot \frac{1}{a}\,du$ $\frac{du}{dx} = a$

$= \frac{1}{a} e^{u} + C$ $dx = \frac{1}{a}\,du$

$= \frac{1}{a} e^{ax} + C$

Example 8.3.3

$\int 6x(4x^2+3)^7\,dx$ Let $u = 4x^2+3$

$= \int 6(4x^2+3)^7\,x\,dx$ $\frac{du}{dx} = 8x$

$= \int 6\,u^7\,\frac{1}{8}\,dx$ $x\,dx = \frac{1}{8}\,du$

$= \frac{6}{8} \cdot \frac{1}{8} u^8 + C$

$= \frac{3}{32}(4x^2+3)^8 + C$

Some write $\int 6x(4x^2+3)^7\,dx$ $d(4x^2+3) = 8x\,dx$

$= \int 6(4x^2+3)^7\,x\,dx$ $x\,dx = \frac{1}{8}\,d(4x^2+3)$

$= \int 6(4x^2+3)^7 \cdot \frac{1}{8}\,d(4x^2+3)$

$= \frac{6}{8} \cdot \frac{1}{8}(4x+3)^8 + C$

$= \frac{3}{32}(4x^2+3)^8 + C$

Example 8.3.4

$\int \frac{(\ln x)^2}{x}\,dx$, $x > 0$

$\int \frac{(\ln x)^2}{x}\,dx$ Let $u = \ln x$

$= \int u^2\,du$ $\frac{du}{dx} = \frac{1}{x}$

$= \frac{1}{3}u^3 + C$ $\frac{1}{x}\,dx = du$

$= \frac{1}{3}(\ln x)^3 + C$

Question : How to make a guess of $u(x)$?

Integration by Substitution : $\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$

Example : $\int \frac{(\ln x)^2}{x}\,dx = \int (\ln x)^2 \cdot \frac{1}{x}\,dx$ Let $u = \ln x$

Realize the integrand as a product of parts and make a guess of $u(x)$ such that one part can be realized as a function $f(u)$, another part is $u'(x)$.

Exercise 8.3.1

1) Show that $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$.   Hint: Let $u = ax+b$

2) Evaluate

a) $\int x^3 e^{x^4}\,dx$ Hint: Let $u = x^4$ Ans: $\frac{1}{4}e^{x^4} + C$

b) $\int 6x\sqrt{x^2+3}\,dx$ Hint: Let $u = x^2+3$ Ans: $2(x^2+3)^{\frac{3}{2}} + C$

Integration of Expontenial Functions :

Recall: $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

In general: $\int a^x\,dx = ?$ for $a > 0$

Recall: $a^x = e^{\ln a^x} = e^{(\ln a)x}$

$\therefore \int a^x\,dx = \int e^{(\ln a)x}\,dx$

$= \frac{1}{\ln a}e^{(\ln a)x} + C$

$= \frac{a^x}{\ln a} + C$

OR: Recall that $\frac{d}{dx}a^x = a^x \ln a$

so $\frac{d}{dx}\frac{a^x}{\ln a} = a^x$, and $\int a^x\,dx = \frac{a^x}{\ln a} + C$

Integration of Logarithmic Functions :

$\int \ln x\,dx = ?$ for $x > 0$

Exercise: $\frac{d}{dx}\, x\ln x - x$

Ans: $\ln x$!

Therefore, $\int \ln x\,dx = x\ln x - x + C$

Problem: How do we know $\frac{d}{dx}\, x\ln x - x = \ln x$ in advance?

(Make a guess of antiderivative of $\ln x$ directly)

Any direct way to find an antiderivative of $\ln x$? (Yes, later !)

Example 8.3.5   (Constant issue)

$\int (x+1)^2\,dx$ let $u = x+1$

$= \int u^2\,du$ $du = dx$

$= \frac{1}{3}u^3 + C$

$= \frac{1}{3}(x+1)^3 + C$

$= \frac{1}{3}x^3 + x^2 + x + \frac{1}{3} + C$

$\int (x+1)^2\,dx$

$= \int x^2+2x+1\,dx$

$= \frac{1}{3}x^3 + x^2 + x + C$

seems to be different !

Ans: This C is NOT that C !

Integration of Rational Functions :

Rational Functions : a quotient of two polynomials

Rational Function $\longrightarrow$ $R(x) = \dfrac{p(x)}{q(x)}$ $\gtrless$ polynomials

Simplest case : $\deg q(x) = 1$ i.e. $q(x) = ax+b$ where $a \neq 0$.

- $\displaystyle\int \frac{p(x)}{ax+b}\,dx$

$$\begin{array}{r} u(x) \\ ax+b\,\overline{)\,p(x)} \\ \hline R \end{array}$$

By long division, $p(x) = (ax+b)\,u(x) + R$

$$\frac{p(x)}{ax+b} = u(x) + \frac{R}{ax+b}$$

Then $\displaystyle\int \frac{p(x)}{ax+b}\,dx = \int u(x) + \frac{R}{ax+b}\,dx$

We know how to integrate !

Example 8.3.6

$$\begin{aligned} &\int \frac{x^2+3x+5}{x+1}\,dx \\ &= \int x+2+\frac{3}{x+1}\,dx \\ &= \frac{x^2}{2} + 2x + 3\ln|x+1| + C \end{aligned}$$

$$\begin{array}{r} x+2 \\ x+1\,\overline{)\,x^2+3x+5} \\ \underline{x^2+x} \\ 2x+5 \\ \underline{2x+2} \\ 3 \end{array}$$

$\therefore\ x^2+3x+5 = (x+1)(x+2)+3$

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}$$

Exercise : Evaluate $\displaystyle\int \frac{6x^2-5x+1}{3x-2}\,dx$

Ans : $x^2 - \frac{x}{3} + \frac{1}{9}\ln|3x-2| + C$

Next case : $\deg q(x) = 2$ i.e. $q(x) = ax^2+bx+c$ where $a \neq 0$.

If $\deg p(x) \geq 2$, by long division, $\displaystyle\int \frac{p(x)}{ax^2+bx+c}\,dx = \int \underset{\substack{\uparrow \\ \text{polynomial}}}{u(x)} + \frac{rx+s}{ax^2+bx+c}\,dx$

$$\begin{array}{r} u(x) \\ ax^2+bx+c\,\overline{)\,p(x)} \\ \hline rx+s \end{array}$$

Just focus on $\displaystyle\int \frac{rx+s}{ax^2+bx+c}\,dx$

Recall : $\Delta = b^2 - 4ac$

We further consider 3 subcases :

(i) $\Delta > 0$ (ii) $\Delta = 0$ (iii) $\Delta < 0$

(i) $\Delta > 0$, $g(x) = ax^2+bx+c = (m_1x+n_1)(m_2x+n_2)$

Express $\frac{rx+s}{ax^2+bx+c}$ into the form $\frac{A}{m_1x+n_1}+\frac{B}{m_2x+n_2}$.

Then $\int \frac{rx+s}{ax^2+bx+c}\,dx = \int \frac{A}{m_1x+n_1}+\frac{B}{m_2x+n_2}\,dx$

We know how to integrate !

Example 8.3.7

$\int \frac{5x-7}{x^2-2x-3}\,dx$

Note : $\frac{5x-7}{x^2-2x-3} = \frac{5x-7}{(x-3)(x+1)}$

Suppose $\frac{5x-7}{(x-3)(x+1)} \equiv \frac{A}{x-3}+\frac{B}{x+1}$

$\Rightarrow \quad 5x-7 \equiv A(x+1)+B(x-3)$

$\Rightarrow \quad A=2 \ , \ B=3$.

$$\int \frac{5x-7}{x^2-2x-3}\,dx = \int \frac{2}{x-3}+\frac{3}{x+1}\,dx = 2\ln|x-3|+3\ln|x+1|+C$$

Exercise : Evaluate $\int \frac{40}{x(200-x)}\,dx$

Ans : $\frac{1}{5}(\ln|x| - \ln|200-x|)+C = \frac{1}{5}\ln\left|\frac{x}{200-x}\right|+C$

(ii) $\Delta = 0$, $g(x) = ax^2+bx+c = (mx+n)^2$

Express $\frac{rx+s}{ax^2+bx+c}$ into the form $\frac{A}{(mx+n)^2}+\frac{B}{mx+n}$.

Then $\int \frac{rx+s}{ax^2+bx+c}\,dx = \int \frac{A}{(mx+n)^2}+\frac{B}{mx+n}\,dx$

We know how to integrate !

Example 8.3.8

$\int \frac{2x-1}{(x-2)^2}\,dx$

Suppose $\frac{2x-1}{(x-2)^2} \equiv \frac{A}{(x-2)^2}+\frac{B}{x-2}$

$\Rightarrow 2x-1 \equiv A+B(x-2)$

$\Rightarrow A=3 \ , \ B=2$

$$\int \frac{2x-1}{(x-2)^2}\,dx = \int \frac{3}{(x-2)^2}+\frac{2}{x-2}\,dx = \frac{-3}{x-2}+2\ln|x-2|+C$$

Exercise : Evaluate $\int \frac{4x+2}{(2x-1)^2}\,dx$

Ans : $\frac{-2}{2x-1}+\ln|2x-1|+C$

(iii) $\Delta < 0$, $q(x) = ax^2+bx+c$ cannot be factorized as a product of two linear factors

$\int \frac{1}{x^2+a^2}\,dx$ let $x = au$

$= \int \frac{1}{a^2u^2+a^2}\,a\,du$ $dx = a\,du$

$= \frac{1}{a}\int \frac{1}{u^2+1}\,du$

$= \frac{1}{a}\tan^{-1}u + C$

$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

Example 8.3.9

$\int \frac{1}{x^2+2x+5}\,dx$

$= \int \frac{1}{(x+1)^2+2^2}\,dx$ (Completing square)

$= \int \frac{1}{u^2+2^2}\,du$ let $u = x+1$ (or let $x+1 = 2t$, what happens ?)

$= \frac{1}{2}\tan^{-1}\frac{u}{2} + C$ $du = dx$

$= \frac{1}{2}\tan^{-1}\frac{x+1}{2} + C$

Example 8.3.10

$\int \frac{4x+7}{x^2+2x+5}\,dx$ Note : $d(x^2+2x+5) = (2x+2)\,dx$

$= \int \frac{2(2x+2)+3}{x^2+2x+5}\,dx$ and $4x+7 = 2(2x+2)+3$

$= 2\int \frac{2x+2}{x^2+2x+5}\,dx + 3\int \frac{1}{x^2+2x+5}\,dx$

$= 2\ln(x^2+2x+5) + 3\left(\frac{1}{2}\tan^{-1}\frac{x+1}{2}\right) + C$

$= 2\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\frac{x+1}{2} + C$

Integration of Trigonometric Functions :

- $\int \tan x\,dx$ and $\int \cot x\,dx$

$\int \tan x\,dx$

$= \int \frac{\sin x}{\cos x}\,dx$ let $u = \cos x$

$= \int -\frac{1}{u}\,du$ $\frac{du}{dx} = -\sin x$

$= -\ln|u| + C$ $-du = \sin x\,dx$

$= -\ln|\cos x| + C$

$= \ln|\sec x| + C$

$\int \cot x \, dx$

$= \int \frac{\cos x}{\sin x} \, dx$ let $u = \sin x$

Ex : :

$= \ln|\sin x| + C$

· $\int \sec x \, dx$ and $\int \csc x \, dx$, t - substitution

t - substitution

Let $t = \tan \frac{x}{2}$

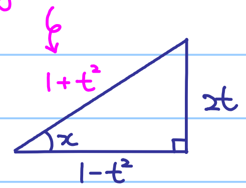
Idea : We can express all trigonometric functions in terms of t.

Note : $\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2t}{1-t^2}$ and so $\cot x = \frac{1-t^2}{2t}$

$\therefore$ $\sin x = \frac{2t}{1+t^2}$ and so $\csc x = \frac{1+t^2}{2t}$

$\cos x = \frac{1-t^2}{1+t^2}$ $\sec x = \frac{1+t^2}{1-t^2}$

By Pyth. thm.



Therefore, all trigonometric functions in terms of t.

Note : $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}(1+t^2)$

$dx = \frac{2}{1+t^2} dt$

Idea : $\int f(\sin x, \cos x) \, dx$

$= \int \underbrace{f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2}{1+t^2}}_{\text{Rational functions of } t.} dt$

Transforming an integral of trigonometric function to an integral of rational function.

$\int \csc x \, dx$

$= \int \frac{1+t^2}{2t} \frac{2}{1+t^2} dt$

$= \int \frac{1}{t} dt$

$= \ln|t| + C$

$= \ln\left|\tan\frac{x}{2}\right| + C$

$\int \sec x \, dx$

$= \int \frac{1+t^2}{1-t^2} \frac{2}{1+t^2} dt$

$= \int \frac{2}{1-t^2} dt$

$= \int \frac{1}{1+t} + \frac{1}{1-t} \; dt$

$= \ln|1+t| - \ln|1-t| + C$

$= \ln\left|\frac{1+t}{1-t}\right| + C$

$= \ln\left|\frac{2t+1-t^2}{1-t^2}\right| + C$

$= \ln|\tan x + \sec x| + C$

Example 8.3.11

$\int \frac{1}{1+\cos x} dx$

$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt$

$= \int dt$

$= t + C$

$= \tan\frac{x}{2} + C$

Remark: t-substitution is particular useful for $\int \frac{1}{A\cos x + B\sin x + C} dx$.

Exercise 8.3.2

By t-substitution, show that $\int \sin x \, dx = -\frac{2}{1+\tan^2\frac{x}{2}} + C$.

However, we know $\int \sin x \, dx = -\cos x + C$

What is the relation between $-\frac{2}{1+\tan^2\frac{x}{2}}$ and $-\cos x$?

Ans : $-\frac{2}{1+\tan^2\frac{x}{2}} - (-\cos x) = -\frac{2}{1+t^2} + \frac{1-t^2}{1+t^2} = -1$ which is a constant.

Exercise 8.3 2

Show that

a) $\int \sin px \, dx = -\frac{1}{p} \cos px + C$

b) $\int \cos px \, dx = \frac{1}{p} \sin px + C$

- $\int \sin px \cos qx \, dx$, $\int \sin px \sin qx \, dx$, $\int \cos px \cos qx \, dx$

Recall : $\sin px \cos qx = \frac{1}{2}[\sin(p+q)x + \sin(p-q)x]$

$\cos px \cos qx = \frac{1}{2}[\cos(p+q)x + \cos(p-q)x]$

$\sin px \sin qx = -\frac{1}{2}[\cos(p+q)x - \cos(p-q)x]$

We know how to integrate RHS !

Example 8.3.12

$\int \sin 5x \cos 3x \, dx$

$= \frac{1}{2}\int \sin 8x + \sin 2x \, dx$

$= \frac{1}{2}(-\frac{\cos 8x}{8} - \frac{\cos 2x}{2}) + C$

$= -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C$

In particular, $\cos^2 px = \frac{1}{2}(1+\cos 2px)$

$\sin^2 px = \frac{1}{2}(1-\cos 2px)$

Example 8.3.13

$\int \cos x \cos^2 3x \, dx$

$= \int \cos x \, [\frac{1}{2}(1+\cos 6x)] \, dx$

$= \frac{1}{2}\int \cos x \, dx + \frac{1}{2}\int \cos x \cos 6x \, dx$

$= \frac{1}{2}\int \cos x \, dx + \frac{1}{4}\int \cos 7x + \cos 5x \, dx$

$= \frac{\sin x}{2} + \frac{\sin 7x}{28} + \frac{\sin 5x}{20} + C$

Exercise : Find $\int \sin x \sin 3x \sin 6x \, dx$

Ans : $\frac{\cos 10x}{40} + \frac{\cos 2x}{8} - \frac{\cos 8x}{32} - \frac{\cos 4x}{16} + C$

- $\int \sin^m x \cos^n x \, dx$

Case 1 : m is odd

Apply : $\sin x \, dx = -d\cos x$ and $\sin^2 x = 1 - \cos^2 x$

Example 8.3.14

$$\int \sin^3 x \cos^2 x \, dx$$

$$= \int \sin^2 x \sin x \cos^2 x \, dx$$

$$= -\int \sin^2 x \cos^2 x \, d\cos x$$

$$= -\int (1-\cos^2 x) \cos^2 x \, d\cos x$$

$$= \int -\cos^2 x + \cos^4 x \, d\cos x$$

$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

Case 2 : n is odd

Similar to case 1

Apply : $\cos x \, dx = d\sin x$ and $\cos^2 x = 1 - \sin^2 x$

Example 8.3.15

$$\int \sin^4 x \cos^3 x \, dx$$

$$= \int \sin^4 x \cos^2 x \cos x \, dx$$

$$= \int \sin^4 x (1-\sin^2 x) \, d\sin x$$

⋮ Ex

$$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$$

Case 3 : m and n are even.

Apply : $\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin x \cos x = \frac{1}{2}\sin 2x$

Example 8.3.16

$$\int \sin^2 x \cos^4 x \, dx$$

$$= \int (\sin x \cos x)^2 \cos^2 x \, dx$$

$$= \int \left(\frac{1}{4}\sin^2 2x\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{8}\underline{\int \sin^2 2x \, dx} + \frac{1}{8}\underline{\int \sin^2 2x \cos 2x \, dx}$$

reduce to case 1

case 3 again

$$= \frac{1}{16}\int 1 - \cos 4x \, dx + \frac{1}{8}\int \sin^2 2x \, \frac{1}{2} d\sin 2x$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C$$

- $\int \tan^m x \sec^n x \, dx$

Case 1. m is odd

Apply : $\tan x \sec x \, dx = d \sec x$ and $\tan^2 x = 1 - \sec^2 x$

Example 8.3.17

$\int \tan^3 x \sec^4 x \, dx$

$= \int \tan^2 x \tan x \sec^3 x \sec x \, dx$

$= \int \tan^2 x \sec^3 x \, d\sec x$

$= \int (\sec^2 x - 1) \sec^3 x \, d\sec x$

$= \int \sec^5 x - \sec^3 x \, d\sec x$

$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

Case 2. n is even

Similar to case 1

Apply : $\sec^2 x \, dx = d \tan x$ and $\sec^2 x = 1 + \tan^2 x$

Example 8.3.18

$\int \tan^4 x \sec^4 x \, dx$

$= \int \tan^4 x \sec^2 x \sec^2 x \, dx$

$= \int \tan^4 x (1 + \tan^2 x) \, d\tan x$

⋮ Ex

$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$

Case 3. m is even and n is odd

Using integration by parts, later!

- $\int \csc^m x \cot^n x \, dx$

Similarly, apply $\csc^2 x = -d \cot x$

$\csc x \cot x = -d \csc x$

$1 + \cot^2 x = \csc^2 x$

Exercise : Find

a) $\int \csc^6 x \cot^4 x \, dx$ Ans : $-\frac{\cot^9 x}{9} - \frac{2\cot^7 x}{7} - \frac{\cot^5 x}{5} + C$

b) $\int \csc^5 x \cot^3 x \, dx$ $-\frac{\csc^7 x}{7} + \frac{\csc^5 x}{5} + C$

Integration of Irrational Functions :

- Integrand with $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$ $(a>0)$

(1) For $\sqrt{a^2-x^2}$, we let $x = a\sin\theta$ $\quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$

(2) For $\sqrt{a^2+x^2}$, we let $x = a\tan\theta$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(3) For $\sqrt{x^2-a^2}$, we let $x = a\sec\theta$ $\quad 0 \leqslant \theta \leqslant \pi \,,\ \theta \neq \frac{\pi}{2}$

Example 8.3.19

$$\int x^3\sqrt{4-x^2}\,dx$$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$$= \int 8\sin^3\theta\,\sqrt{4\cos^2\theta}\,(2\cos\theta)\,d\theta$$

$$= \int 32\cos^2\theta\sin^3\theta\,d\theta$$

$$= \int 32\cos^2\theta\,\sin^2\theta\,\sin\theta\,d\theta$$

$$= \int 32\cos^2\theta\,(1-\cos^2\theta)\,d(-\cos\theta)$$

$$= \int 32\cos^4\theta - 32\cos\theta\,d\cos\theta$$

$$= \frac{32}{5}\cos^5\theta - \frac{32}{3}\cos^3\theta + C$$

$$= \frac{32}{5}\left(\frac{\sqrt{4-x^2}}{2}\right)^5 - \frac{32}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + C$$

$$= -\frac{1}{15}(3x^2+8)(4-x^2)^{\frac{3}{2}} + C$$

$x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2}$

$\cos\theta = \pm\sqrt{1-\sin^2\theta} = \sqrt{1-(\frac{x}{2})^2} = \pm\frac{\sqrt{4-x^2}}{2}$

$-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2} \Rightarrow \cos\theta > 0$

$\therefore \cos\theta = \frac{\sqrt{4-x^2}}{2}$

Note : $\sqrt{a^2-x^2}$ is well-defined only when $a^2-x^2 \geqslant 0$, that means $-a<x<a$

Also we have $-1 \leqslant \sin\theta \leqslant 1$ when $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$,

so $-a \leqslant a\sin\theta \leqslant a$, that is the reason why we let $x = a\sin\theta$.

Think : How about $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$?

Example 8.3.20

$$\int \frac{\sqrt{x^2-4}}{x^3}\,dx$$

Let $x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$

$$= \int \frac{\sqrt{4\tan^2\theta}}{8\sec^3\theta}\,2\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{2}\int \sin^2\theta\,d\theta$$

$$= \frac{1}{4}\int 1-\cos 2\theta\,d\theta$$

$$= -\frac{1}{8}\sin 2\theta + \frac{\theta}{4} + C$$

$\vdots$ Ex

$$= -\frac{\sqrt{x^2-4}}{2x^2} + \frac{1}{4}\cos^{-1}\frac{2}{x} + C$$

Exercise 8.3 3

Show that, for $a>0$,

a) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) + C$

b) $\int \sqrt{x^2\pm a^2}\,dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \pm \frac{1}{2}a^2\ln\left|x+\sqrt{x^2\pm a^2}\right| + C$

8.4 Integration by Parts

Recall: Let $u(x)$ and $v(x)$ be differentiable functions.

Product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrate both sides with respect to x:

$$\int u\frac{dv}{dx}\,dx = \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

OR: $\int u\,dv = uv - \int v\,du$

Integration by Parts. $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Example 8.4.1

$$\begin{aligned}
\int x^2\ln x\,dx &= \int (\ln x)\,x^2\,dx \\
&= \int (\ln x)\frac{d}{dx}\left(\frac{x^3}{3}\right)dx \qquad (\text{Now, } u=\ln x,\ v=\frac{x^3}{3}) \\
&= \int \ln x\, d\frac{x^3}{3} \\
&= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\,d(\ln x) \\
&= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{1}{x}\,dx \\
&= \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx \\
&= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C
\end{aligned}$$

(Verify the answer by differentiation!)

Example 8.4.2

$\int x e^x dx$ Note : $\frac{d}{dx} e^x = e^x$

$= \int x \, de^x$ $e^x dx = de^x$

$= xe^x - \int e^x dx$ Now, $u = x$, $v = e^x$

$= xe^x - e^x + C$

$= e^x (x-1) + C$

Remark: Why don't we try the following?

$\int xe^x dx$

$= \int e^x \, x dx$

$= \int e^x \, d(\frac{x^2}{2})$

$\vdots$

What happens?

Example 8.4.3

$\int x^2 e^x dx$

$= \int x^2 de^x$

$= x^2 e^x - \int e^x dx^2$

$= x^2 e^x - \underbrace{\int 2xe^x dx}_{\text{Apply Integration by parts again!}}$

Ex :

Ans : $e^x(x^2 - 2x + 2) + C$

Question : How to make a guess of $u(x)$ and $v(x)$?

Integration by Parts : $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

Example : $\int x^2 \ln x \, dx = \int (\ln x) \, x^2 \, dx$

$= \int (\ln x) \frac{d}{dx} (\frac{x^3}{3}) \, dx$

Realize the integrand as a product of parts and make a guess of $u(x)$ and $v(x)$ such that one part can be realized as a function $u(x)$, another part is $v'(x)$

Example 8.4.4

$\int x \sin 3x \, dx$

$= \int x \, d(-\frac{1}{3}\cos 3x)$

$= -\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x \, dx$

$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$

Integration of Logarithmic Functions :

$\int \ln x \, dx = ?$ for $x > 0$

Using Integration by part .

$\int \ln x \, dx = x \ln x - \int x \, d \ln x$ $u = \ln x$ $v = x$

$= x \ln x - \int x \cdot \frac{1}{x} dx$

$= x \ln x - \int dx$

$= x \ln x - x + C$

Exercise 8.4.1

Find $\int \log_a x \, dx$

Hints . $\log_a x = \frac{\ln x}{\ln a}$

$\int \log_a x \, dx = \frac{1}{\ln a} \int \ln x \, dx$

$= \frac{1}{\ln a}(x \ln x - x + C)$

$= x\frac{\ln x}{\ln a} - \frac{x}{\ln a} + \frac{C}{\ln a}$

$= x\log_a x - \frac{x}{\ln a} + C'$ $C' = \frac{C}{\ln a}$ just a constant !

Example 8.4.5 (Transformed into the original Integral)

$$\int e^x \cos x\, dx = \int e^x\, d\sin x$$
$$= e^x \sin x - \int \sin x\, de^x$$
$$= e^x \sin x - \int e^x \sin x\, dx$$
$$= e^x \sin x - \int e^x\, d(-\cos x)$$
$$= e^x \sin x - (-e^x \cos x - \int -\cos x\, de^x)$$
$$= e^x \sin x - (-e^x \cos x - \int -e^x \cos x\, dx)$$
$$= e^x \sin x + e^x \cos x - \underbrace{\int e^x \cos x\, dx}_{\text{back to itself !}}$$

Be careful of +/- !

$$\therefore\ 2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C' \leftarrow \text{Don't forget !}$$
$$\int e^x \cos x\, dx = \tfrac{1}{2} e^x (\sin x + \cos x) + C \qquad (C = \tfrac{1}{2} C')$$

Example 8.4.6

$$\int \sin(\ln x)\, dx$$
$$= x \sin(\ln x) - \int x\, d\sin(\ln x)$$
$$= x \sin(\ln x) - \int \cos(\ln x)\, dx$$
$$= x \sin(\ln x) - (x\cos(\ln x) - \int x\, d\cos(\ln x))$$
$$= x \sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\, dx$$

$$\therefore \int \sin(\ln x)\, dx = \tfrac{1}{2} x [\sin(\ln x) + \cos(\ln x)] + C$$

Exercise 8.4.2

$$\int \sec^3 x\, dx$$
$$= \int \sec x\, (\sec^2 x)\, dx$$
$$= \int \sec x\, d\tan x$$

Ex :

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$\therefore \int \sec^3 x\, dx = \tfrac{1}{2}[\sec x \tan x + \ln|\sec x + \tan x|] + C$$

Think : In general, how to find $\int \tan^m x \sec^n x\, dx$ if m is even, n is odd ?

8.5 Reduction Formulae

Idea : Obtain a formula to reduce the complexity of the integrand.

Example 8.5.1

Let $I_n = \int x^n e^x \, dx$, where n is a nonnegative integer.

Prove that $I_n = x^n e^x - n I_{n-1}$, for $n \geq 1$.

$$
\begin{aligned}
I_n &= \int x^n e^x dx \\
&= \int x^n de^x \\
&= x^n e^x - \int e^x dx^n \\
&= x^n e^x - \int n e^x x^{n-1} dx \\
&= x^n e^x - n I_{n-1}
\end{aligned}
$$

Note : $I_0 = \int e^x dx = e^x + C$

We can apply this formula repeatly until we see I_0 :

$$
\begin{aligned}
\int x^3 e^x dx = I_3 &= x^3 e^x - 3 I_2 \\
&= x^3 e^x - 3(x^2 e^x - 2 I_1) \\
&= x^3 e^x - 3(x^2 e^x - 2(x e^x - 1 \cdot I_0)) \\
&= x^3 e^x - 3x^2 e^x + 3 \cdot 2 x e^x - 3 \cdot 2 \cdot 1 \cdot I_0 \\
&= x^3 e^x - 3x^2 e^x + 3 \cdot 2 x e^x - 3 \cdot 2 \cdot 1 \cdot e^x + C \\
&= x^3 e^x - P^3_1 x^2 e^x + P^3_2 x e^x - P^3_3 e^x + C \\
&= \left[\sum_{r=0}^{3} (-1)^r P^3_r x^{3-r} e^x \right] + C
\end{aligned}
$$

In general, $\int x^n e^x dx = \left[\sum_{r=0}^{n} (-1)^r P^n_r x^{n-r} e^x \right] + C$ for $n \geq 1$.

The formula $I_n = x^n e^x - n I_{n-1}$ is called a reduction formula.

Example 8.5.2

Let $I_n = \int \tan^n x \, dx$, where n is a nonnegative integer.

Show that $I_n = \underbrace{\frac{1}{n-1}\tan^{n-1}x}_{} - I_{n-2}$ for $n \geq 2$.

Why / How do we get this?

$\int \tan^{n-2}x \, d\tan x$

$$
\begin{aligned}
I_n &= \int \tan^n x \, dx \\
&= \int \tan^{n-2}x \tan^2 x \, dx \\
&= \int \tan^{n-2}x \, (\sec^2 x - 1) \, dx \\
&= \int \tan^{n-2}x \sec^2 x \, dx - \int \tan^{n-2}x \, dx \\
&= \int \tan^{n-2}x \, d\tan x - I_{n-2} \\
&= \frac{1}{n-1}\tan^{n-1}x - I_{n-2}
\end{aligned}
$$

As we can see, the index n is decreased by 2 when the reduction formula is applied, so we have two cases:

Case 1: start from an even integer $n = 2m$

$$
\begin{aligned}
I_{2m} &= \frac{1}{2m-1}\tan^{2m-1}x - I_{2m-2} \\
&= \frac{1}{2m-1}\tan^{2m-1}x - \frac{1}{2m-3}\tan^{2m-3}x + I_{2m-4} \\
&\ \vdots \\
&= \frac{1}{2m-1}\tan^{2m-1}x - \frac{1}{2m-3}\tan^{2m-3}x + \cdots + (-1)^{m-2}\frac{1}{3}\tan^3 x + (-1)^{m-1}\tan x + (-1)^m I_0 \quad \text{(end at } I_0\text{)} \\
&= \frac{1}{2m-1}\tan^{2m-1}x - \frac{1}{2m-3}\tan^{2m-3}x + \cdots + (-1)^{m-2}\frac{1}{3}\tan^3 x + (-1)^{m-1}\tan x + (-1)^m x + C \quad \left(I_0 = \int dx = x + C\right)
\end{aligned}
$$

Case 2: start from an odd integer $n = 2m+1$

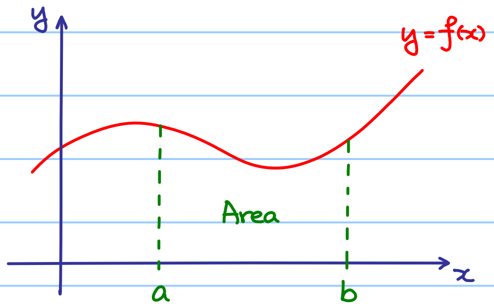
$$
\begin{aligned}
I_{2m+1} &= \frac{1}{2m}\tan^{2m}x - I_{2m-1} \\
&= \frac{1}{2m}\tan^{2m}x - \frac{1}{2m-2}\tan^{2m-2}x + I_{2m-3} \\
&\ \vdots \\
&= \frac{1}{2m}\tan^{2m}x - \frac{1}{2m-2}\tan^{2m-2}x + \cdots + (-1)^{m-2}\frac{1}{4}\tan^4 x + (-1)^{m-1}\frac{1}{2}\tan^2 x + (-1)^m I_1 \quad \text{(end at } I_1\text{)} \\
&= \frac{1}{2m}\tan^{2m}x - \frac{1}{2m-2}\tan^{2m-2}x + \cdots + (-1)^{m-2}\frac{1}{4}\tan^4 x + (-1)^{m-1}\frac{1}{2}\tan^2 x + (-1)^m \ln|\sec x| + C
\end{aligned}
$$

$\left(I_1 = \int \tan x \, dx = \ln|\sec x| + C\right)$

§ 9 Definite Integration

9.1 Riemann Sum

Goal : Find the area of the region under the curve $y=f(x)$ over an interval $[a,b]$.



Wait! We know what the area of a rectangle is.

However, what is the area of a region with a curved boundary? (How to define ?)

Idea :

Approximate by rectangles !

A partition of the interval $[a,b]$ is a finite set $\{x_0, x_1, \cdots, x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b\ .$$

We denote $\Delta x_k = x_k - x_{k-1}$ for $k=1,2,\cdot\ ,n$.

Then, we choose points, $c_1, c_2, \cdots, c_n$, called partition points so that

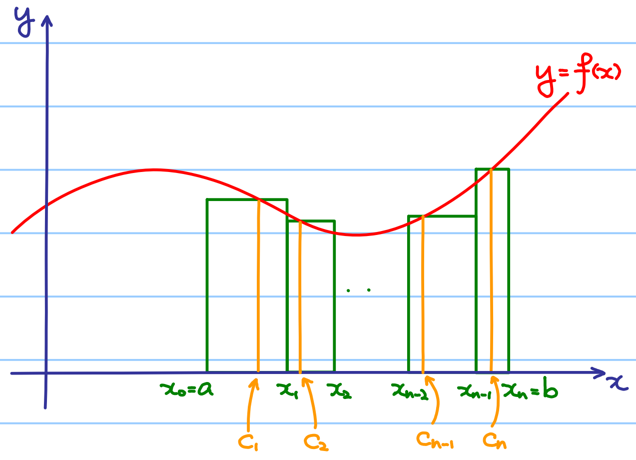
$$x_{k-1} \leq c_k \leq x_k \quad \text{for } k=1,2,\cdots,n.$$

Definition 9.1.1

Let $f:[a,b]\to\mathbb{R}$. The Riemann sum is defined by $\sum_{k=1}^{n} f(c_k)\Delta x_k$.

In particular, if $x_{k-1}=c_k$, the sum is called the left Riemann sum ;

if $c_k = x_k$, the sum is called the right Riemann sum.



For the k-th rectangle :

$$\underbrace{f(c_k)}_{\text{height}}\ \times\ \underbrace{\Delta x_k}_{\text{width}} = \text{area of the k-th rectangle}$$

Example 9.1.1

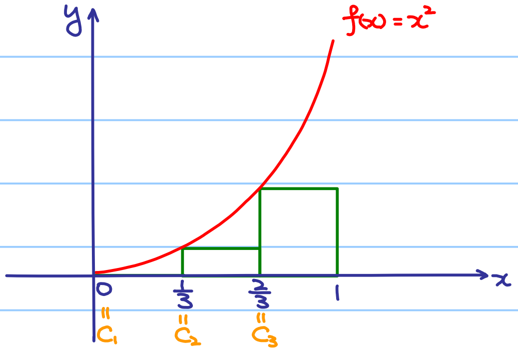
Let $f(x)=x^2$.

Approximate area under $f(x)$ over $[0,1]$

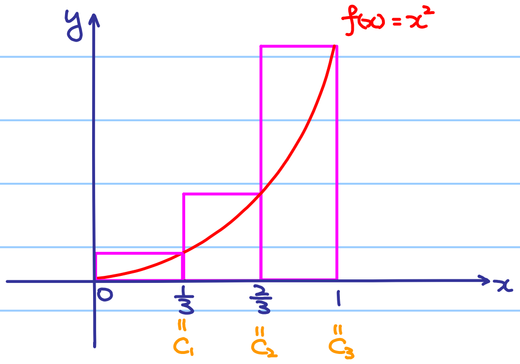
with 3 even partitions : $0<\frac{1}{3}<\frac{2}{3}<1$ $(x_0=0\ ,\ x_1=\frac{1}{3}\ ,\ x_2=\frac{2}{3}\ ,\ x_3=1)$

Riemann Sum :

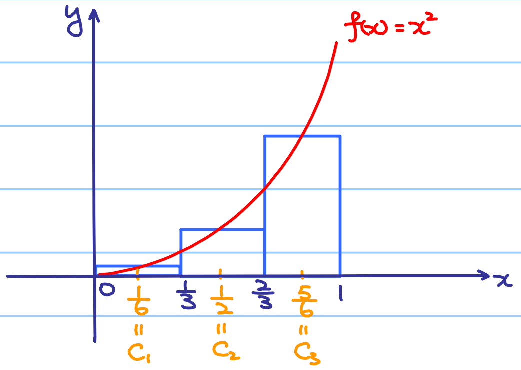
Left sum : $c_1=0\ ,\ c_2=\frac{1}{3}\ ,\ c_3=\frac{2}{3}$ area ≈ $0^2\cdot\frac{1}{3}+(\frac{1}{3})^2\cdot\frac{1}{3}+(\frac{2}{3})^2\cdot\frac{1}{3}=\frac{5}{27}$



Right sum : $c_1=\frac{1}{3}\ ,\ c_2=\frac{2}{3}\ ,\ c_3=1$ area ≈ $(\frac{1}{3})^2\cdot\frac{1}{3}+(\frac{2}{3})^2\cdot\frac{1}{3}+1^2\ \frac{1}{3}=\frac{14}{27}$



Mid-pt sum : $c_1=\frac{1}{6}\ ,\ c_2=\frac{1}{2}\ ,\ c_3=\frac{5}{6}$ area ≈ $(\frac{1}{6})^2\cdot\frac{1}{3}+(\frac{1}{2})^2\cdot\frac{1}{3}+(\frac{5}{6})^2\cdot\frac{1}{3}=\frac{35}{108}$



Idea : Increasing n (number of rectangles) ⇒ Better approximation

Theorem 9.1.1

If $f:[a,b]\to\mathbb{R}$ is continuous (or piecewise continuous),

and $\Delta x_k = \Delta x = \frac{b-a}{n}$ for $k=1,2,\cdots,n$ (even partition), $x_k = a+k\Delta x$ for $k=0,1,2,\cdots,n$,

then no matter how c_k' are chosen, $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} f(c_k)\Delta x$ is always the same

The area under $f(x)$ over $[a,b]$ is defined to be this number,

which is denoted by $\int_a^b f(x)\,dx$.

(Remark: Nothing related to indefinite integration so far!)

In particular,

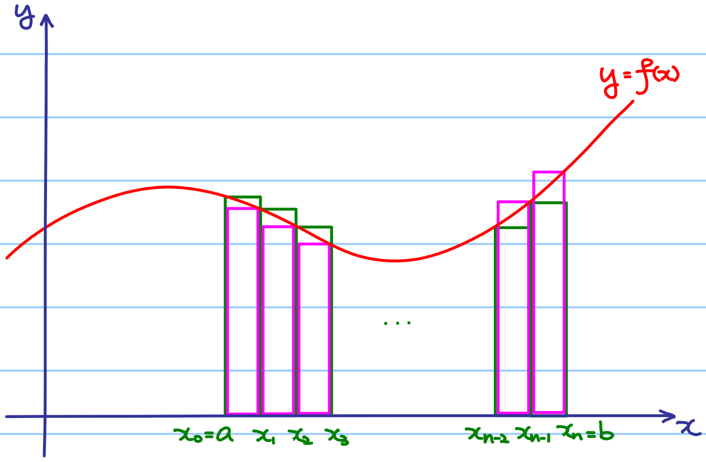
$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_{k-1})\Delta x \qquad (\text{take } c_k = x_{k-1})$$

$$= \lim_{n\to\infty}[f(x_0)+f(x_1)+\cdots+f(x_{n-1})]\Delta x$$

$$\left(= \lim_{n\to\infty}\sum_{k=0}^{n-1} f(x_k)\Delta x \right)$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x \qquad (\text{take } c_k = x_k)$$

$$= \lim_{n\to\infty}[f(x_1)+f(x_2)+\cdots+f(x_n)]\Delta x$$



Example 9.1.2

Let $f(x)=x$, for $0\le x\le 1$. (Take $a=0$, $b=1$, $\Delta x = \frac{b-a}{n} = \frac{1}{n}$, $x_k = a+k\Delta x = \frac{k}{n}$)

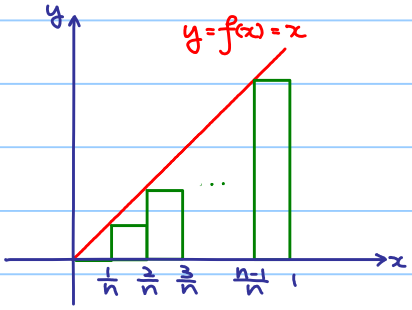
$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_{k-1})\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{k-1}{n}\right)\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k-1}{n}\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\left(\sum_{k=1}^{n} k-1\right)$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\left[\frac{n(n-1)}{2}\right]$$

$$= \frac{1}{2}$$



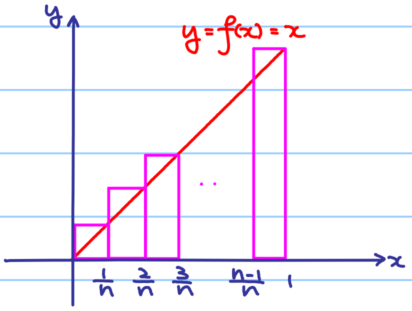
$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n}\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\left(\sum_{k=1}^{n} k\right)$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{1}{2}$$



In fact, computation of the area is not relying on the above theorem,

but the fundamental theorem of calculus (Later!)

9.2 Rules for Definite Integration

Theorem 9.2.1

Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous (or piecewise continuous) functions. Suppose $a \leq b$.

1) If m is a constant, $\int_a^b m f(x)dx = m \int_a^b f(x)dx$
2) $\int_a^b f(x) \pm g(x)\, dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
3) $\int_a^a f(x)dx = 0$
4) $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$ (reverse direction)
5) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for any $c \in \mathbb{R}$ (subdivision)

Idea of proof:

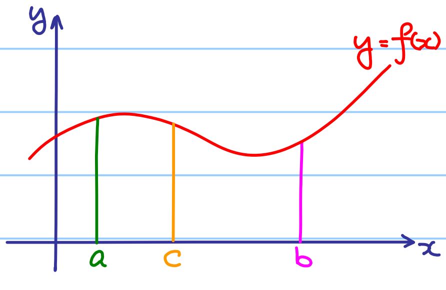
1) $\int_a^b m f(x)dx = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} m f(c_k)\Delta x = m \left(\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(c_k)\Delta x \right) = m \int_a^b f(x)dx$

2) $\int_a^b f(x) \pm g(x)\, dx = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} [f(c_k) \pm g(c_k)] = \lim\limits_{n\to\infty} \left(\sum\limits_{k=1}^{n} f(c_k)\Delta x \pm \sum\limits_{k=1}^{n} g(c_k)\Delta x \right) = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

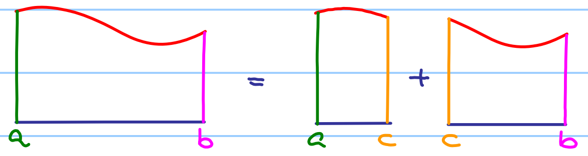
3) $\int_a^a f(x)dx =$ Area of a line segment $= 0$

4) $\int_b^a f(x)\, dx = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(c_k)\Delta x = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(c_k) \frac{a-b}{n} = -\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(c_k) \frac{b-a}{n} = -\int_a^b f(x)\, dx$

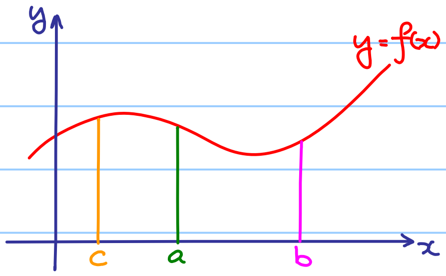
5) If $a \leq c \leq b$,



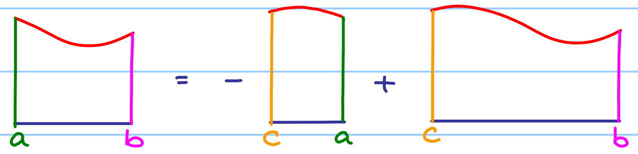
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$



If $c < a \leq b$,



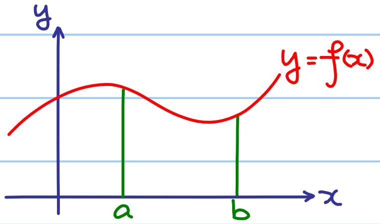
$$\int_a^b f(x)dx = \underbrace{\int_a^c f(x)dx}_{= -\int_c^a f(x)dx} + \int_c^b f(x)dx$$



Exercise: Why is (5) true if $a \leq b < c$?

Theorem 9.2.2

If $f:[a,b]\longrightarrow\mathbb{R}$ is a continuous function such that $f(x)\geq 0$ for all $x\in[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.



Idea of proof:

$$\int_b^a f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\,\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} \underset{\substack{\vee\!|\\ 0}}{f(c_k)}\ \underset{\substack{\vee\!|\\ 0}}{\frac{b-a}{n}} \geq 0$$

Corollary 9.2.1

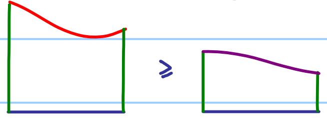
If $f, g:[a,b]\longrightarrow\mathbb{R}$ are continuous functions such that $f(x)\geq g(x)$ for all $x\in[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

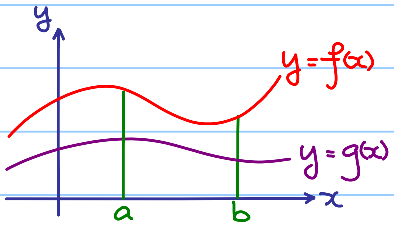
proof:

$f(x)-g(x)\geq 0$ for all $x\in[a,b]$

$$\Rightarrow \int_a^b f(x)-g(x)\,dx \geq 0$$

$$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$



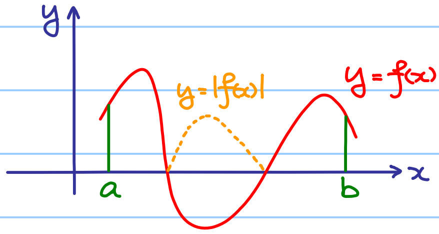


Corollary 9.2.2

If $f:[a,b]\longrightarrow\mathbb{R}$ is a continuous function, then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

(i.e. $-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$)



proof:

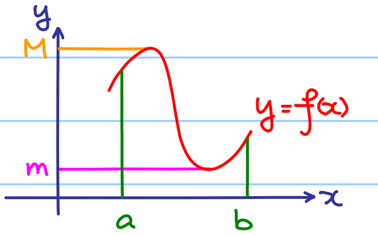
Note: $-|f(x)|\leq f(x)\leq |f(x)|$ for all $x\in[a,b]$

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

Corollary 9.2.3

If $f:[a,b]\longrightarrow\mathbb{R}$ is a continuous function such that $m \le f(x) \le M$ for all $x\in[a,b]$, where $m, M \in \mathbb{R}$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

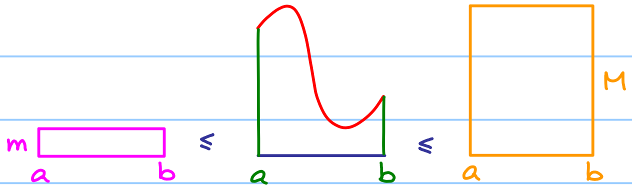


proof:

$$m \le f(x) \le M \quad \text{for all } x\in[a,b]$$

$$\Rightarrow \int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

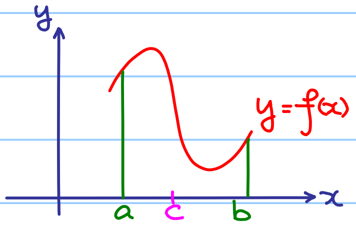
$$\Rightarrow m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

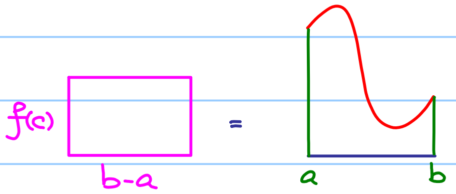


Theorem 9.2.3 (Mean Value Theorem for integrals)

Let $f:[a,b]\rightarrow\mathbb{R}$ be a continuous function

Then, there exists $c\in[a,b]$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$





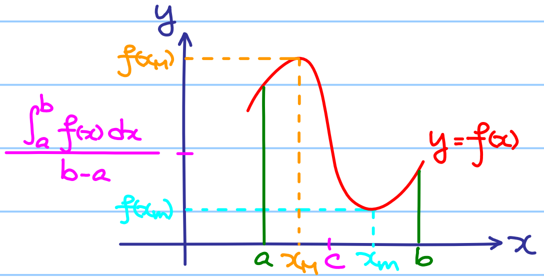
proof:

Since f is continuous on $[a,b]$, by the Maximum-Minimum Theorem (theorem 4.5.1), there exist $x_m, x_M \in [a,b]$ such that $f(x_m) \le f(x) \le f(x_M)$ for all $x\in[a,b]$

By corollary 9.2.3,

$$f(x_m)(b-a) \le \int_a^b f(x)\,dx \le f(x_M)(b-a)$$

$$f(x_m) \le \frac{\int_a^b f(x)\,dx}{b-a} \le f(x_M)$$



By the intermediate value theorem (theorem 4.4.1)

there exists c that lies between x_m and x_M such that

$$f(c) = \frac{\int_a^b f(x)\,dx}{b-a}$$

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

9.3 Fundamental Theorem of Calculus

Preparation :

Let $f(t)$, $t\in\mathbb{R}$, be a continuous function.

1) $\int_{x_0}^{x} f(t)dt$ is well defined for all $x\in\mathbb{R}$

2) What is a function ?

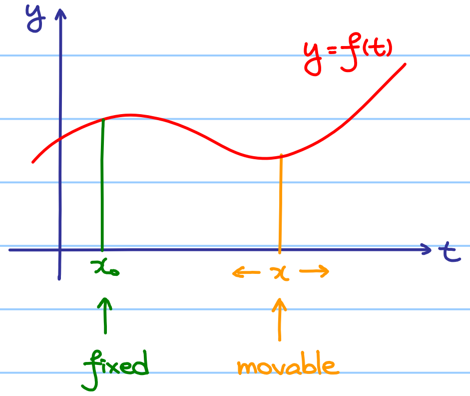
Roughly speaking, input x, output y.

Now, we define

$F(x)$ = Area under the curve $y=f(t)$ over $[x_0, x]$

$= \int_{x_0}^{x} f(t)dt$

What is the relation between $F(x)$ and $f(x)$?



Example 9.3.1

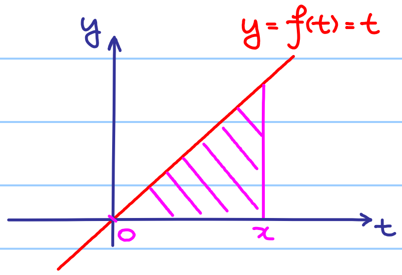
Let $f(t)=t$, $x_0=0$.

$F(x)=\int_0^x f(t)dt$

$=$ Area of the shaded triangle $=\frac{1}{2}x^2$

Note : We have $F'(x)=f(x)$.

Surprisingly, $F(x)$ is an antiderivative of $f(x)$!!!

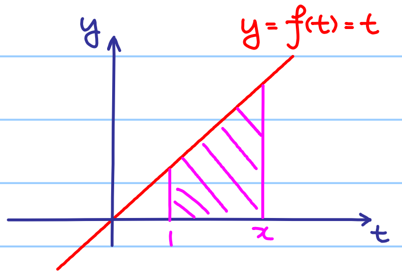


How about fix another point $x_0=1$?

$G(x)=\int_1^x f(t)dt$

$=$ Area of the shaded Trapezium $=\frac{(x+1)(x-1)}{2}=\frac{x^2}{2}-\frac{1}{2}$

Note : We have $G'(x)=f(x)$.



$F(x)-G(x)=\int_0^x f(t)dt-\int_1^x f(t)dt=\int_0^1 f(t)dt$ which is a constant

($=\frac{1}{2}$ which is the area of a triangle)

i.e. $F(x)=G(x)+$ constant

Therefore, if we have $F'(x)=f(x)$, it is natural to have $G'(x)=f(x)$.

It suggests that if $F(x)=\int_{x_0}^{x} f(t)dt$, then $F'(x)=f(x)$ i.e $F(x)$ is an antiderivative of $f(x)$.

Also, different choices of x_0 will give different antiderivatives of $f(x)$, but they just differ by a constant

Theorem 9.3.1 (Fundamental Theorem of Calculus)

Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function and let $x_0 \in \mathbb{R}$.

Suppose $F(x)$ is a function defined by

$$F(x) = \int_{x_0}^{x} f(t)\,dt,$$

then $F(x)$ is a differentiable function and $F'(x) = f(x)$.

(i.e. $F(x)$ is an antiderivative of $f(x)$.)

If we know how to compute an antiderivative $\tilde{F}(x)$ of $f(x)$, then $F(x) = \tilde{F}(x) + C$

Direct consequence:

$$\int_a^b f(x)\,dx = \int_{x_0}^{b} f(x)\,dx - \int_{x_0}^{a} f(x)\,dx$$

$$= \int_{x_0}^{b} f(t)\,dt - \int_{x_0}^{a} f(t)\,dt$$

$$= F(b) - F(a)$$

$$= (\tilde{F}(b) + C) - (\tilde{F}(a) + C)$$

$$= \tilde{F}(b) - \tilde{F}(a) \quad \text{for any antiderivative } \tilde{F}(x) \text{ of } f(x).$$

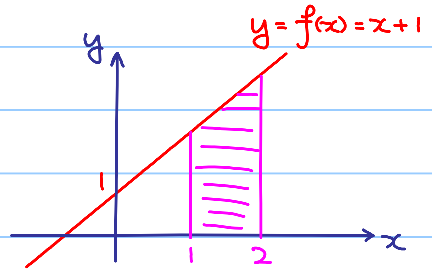
That means if we know how to compute an antiderivative $\tilde{F}(x)$ of $f(x)$, then we can compute the area under the graph of $f(x)$ over $[a,b]$.

Example 9.3.2

Let $f(x) = x+1$

Antiderivative of $f(x) = \int x+1\,dx = \frac{x^2}{2} + x + C$

Choose $C = 0$, let $F(x) = \frac{x^2}{2} + x$



Area of the shaded region $= \int_1^2 f(x)\,dx = F(2) - F(1)$

$$= 4 - \frac{3}{2}$$

$$= \frac{5}{2}$$

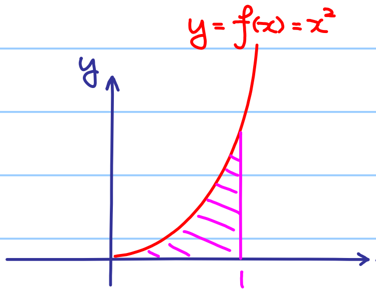
What we write:

$$\int_1^2 f(x)\,dx = \left[\frac{x^2}{2} + x\right]_1^2$$

$$= \underbrace{\left(\frac{2^2}{2} + 2\right)}_{F(2)} - \underbrace{\left(\frac{1^2}{2} + 1\right)}_{F(1)} = 4 - \frac{3}{2} = \frac{5}{2}$$

Example 9.3.3

Let $f(x) = x^2$



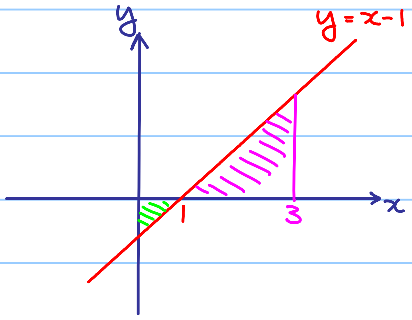
Area of the shaded region $= \int_0^1 f(x)\,dx$

$$= \left[\frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{1^3}{3}\right) - \left(\frac{0^3}{3}\right)$$

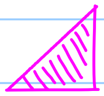
$$= \frac{1}{3}$$

Example 9.3.4 (NOT area, but signed area)



$\int_0^1 x-1\,dx = \left[\frac{x^2}{2} - x\right]_0^1 = -\frac{1}{2}$

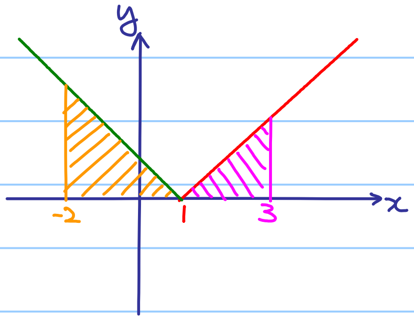
$\int_1^3 x-1\,dx = \left[\frac{x^2}{2} - x\right]_1^3 = 2$

$\int_0^3 x-1\,dx = \left[\frac{x^2}{2} - x\right]_0^3 = \frac{3}{2}$  –  (Cancellation)

Example 9.3.5

Find $\int_{-2}^3 |x-1|\,dx$

Recall : We can rewrite
$$|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ -(x-1) & \text{if } x < -1 \end{cases}$$



$$\int_{-2}^3 |x-1|\,dx = \int_{-2}^1 |x-1|\,dx + \int_1^3 |x-1|\,dx$$

$$= \int_{-2}^1 -(x-1)\,dx + \int_1^3 x-1\,dx$$

Exercise :

$$= \frac{9}{2} + 2 = \frac{13}{2}$$

Example 9.3.6

Find $\frac{dF}{dx}$ if

a) $F(x) = \int_0^x e^{\cos t}\,dt$, b) $F(x) = \int_0^{x^2} e^{\cos t}\,dt$, c) $F(x) = \int_x^{x^2} e^{\cos t}\,dt$

a) $\frac{dF}{dx} = e^{\cos x}$ (Directly from the Fundamental Theorem of Calculus , $f(x) = e^{\cos x}$)

b) $\frac{dF}{dx} = \frac{d}{dx^2}\int_0^{x^2} e^{\cos t}\,dt \quad \frac{dx^2}{dx}$ (Chain rule)

$= e^{\cos x^2} \cdot 2x$

$= 2x e^{\cos x^2}$

c) $\frac{dF}{dx} = \frac{d}{dx}\int_0^{x^2} e^{\cos t}\,dt - \frac{d}{dx}\int_0^{x} e^{\cos t}\,dt$

$= 2xe^{\cos x^2} - e^{\cos x}$

Example 9.3.7

Find $\lim\limits_{n\to\infty} \underbrace{\frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\cdots+\frac{n^2}{n^3}}_{n\text{ terms}} = \lim\limits_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^3}$

Note: As $n\to\infty$, it is an infinite sum , i.e. summing infinitely many terms.

Algebraic rule does NOT work !!

We cannot say: $\lim\limits_{n\to\infty}\frac{1^2}{n^3} = \lim\limits_{n\to\infty}\frac{2^2}{n^3} = \cdots = \lim\limits_{n\to\infty}\frac{n^2}{n^3} = 0$

$\therefore \lim\limits_{n\to\infty} \frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\cdots+\frac{n^2}{n^3} = 0$

Idea:

Regard the infinite sum as the left or right Riemann sum of some function,

so the infinite sum is just the area under that function over an interval.

Recall: area under $f(x)$ over $[a,b] = \int_a^b f(x)dx$

$= \lim\limits_{n\to\infty}\sum_{k=0}^{n-1} f(x_k)\,\Delta x$ (Left)

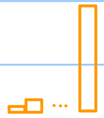
$= \lim\limits_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x$ (Right)

where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$

In this case, take $a=0$, $b=1$,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^3}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^2}\frac{1}{n}$$

$$=\lim_{n\to\infty}\underbrace{\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\frac{1}{n}}$$

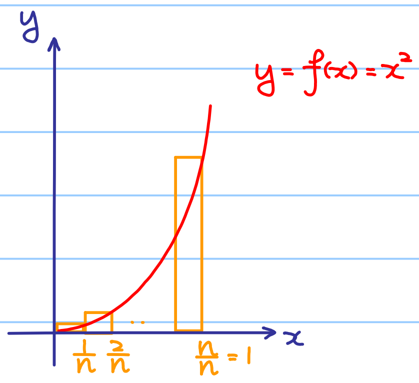


$$=\int_0^1 f(x)\,dx$$

$$=\int_0^1 x^2\,dx$$

$$=\left[\frac{1}{3}x^3\right]_0^1$$

$$=\frac{1}{3}$$



Right Riemann sum

Example 9.3.8

Find $\lim_{n\to\infty}\frac{1}{n}\left(e^{1/n}+e^{2/n}+e^{3/n}+\cdots+e^{n/n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}e^{i/n}$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}e^{i/n}=\lim_{n\to\infty}\sum_{i=1}^{n}e^{i/n}\frac{1}{n}$$

$$=\int_0^1 e^x\,dx$$

$$=\left[e^x\right]_0^1$$

$$=e^1-e^0$$

$$=e-1$$

Example 9.3.10

Find $\lim_{n\to\infty}\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+(n-1)^2}=\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{n}{n^2+k^2}$

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{n}{n^2+k^2}=\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{1+\left(\frac{k}{n}\right)^2}\cdot\frac{1}{n}$$

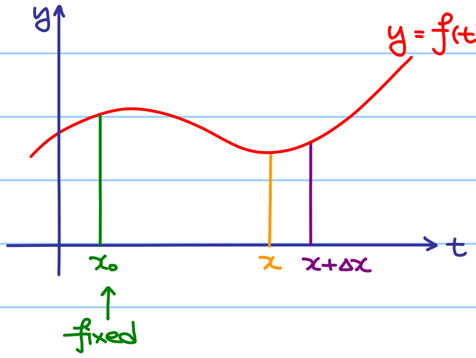
$$=\int_0^1\frac{1}{1+x^2}\,dx$$

$$=\left[\tan^{-1}x\right]_0^1$$

$$=\frac{\pi}{4}$$

Proof of the Fundamental Theorem of Calculus :

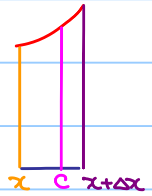
Claim : If $F(x) = \int_{x_0}^{x} f(t)\,dt$, $\lim_{\Delta x \to 0} \frac{F(x+\Delta x)-F(x)}{\Delta x} = f(x)$, i.e. $F'(x) = f(x)$



$F(x+\Delta x) - F(x)$

$= \int_{x}^{x+\Delta x} f(t)\,dt$

= area of



$= f(c)\,\Delta x$ for some c between x and x+Δx

(Mean Value theorem for integrals)

$\lim_{\Delta x \to 0} \frac{F(x+\Delta x)-F(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{f(c)\,\Delta x}{\Delta x}$

$= \lim_{\Delta x \to 0} f(c)$

$= \lim_{c \to x} f(c)$ (As Δx tends to 0 , c tends to x)

$= f(x)$ (By continunity of f)

$\therefore$ $F(x)$ is differentiable and $F'(x) = f(x)$.

9.4 Definite Integral Using Substitution

Theorem 9.4.1

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

Example 9.4.1

Evaluate $\int_0^1 8x(x^2+1)\,dx$

$\int_0^1 8x(x^2+1)\,dx$

$= \int_1^2 8u \,\frac{1}{2} du$

$= \int_1^2 4u\,du$

$= [2u^2]_1^2$

$= 6$

let $u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{1}{2} du = x\,dx$

} Similar to indefinite integration

when $x = 0$, $u = 1$

$x = 1$, $u = 2$

} New ! Don't forget !

Remark:

Some may write :

Still 0 and 1

$$\int_0^1 8x(x^2+1)\,dx = \int_0^1 4(x^2+1)\,d(x^2+1) \qquad (\text{as } d(x^2+1) = 2x\,dx)$$

$$= \left[2(x^2+1)\right]_0^1$$

$$= 6$$

(Just the same result !)

Example 9.4.2

Evaluate $\int_e^{e^2} \frac{1}{x\ln x}\,dx$

$$\int_e^{e^2} \frac{1}{x\ln x}\,dx$$

$$= \int_1^2 \frac{1}{u}\,du$$

$$= [\ln u]_1^2$$

$$= \ln 2 - \cancelto{0}{\ln 1}$$

$$= \ln 2$$

Let $u = \ln x$

$du = \frac{1}{x}dx$

when $x = e$, $u = 1$

$x = e^2$, $u = 2$

Recall :

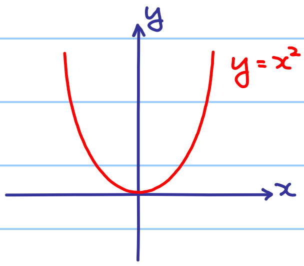
Definition 9 4.1

A function $f: \mathbb{R} \longrightarrow \mathbb{R}$ is said to be

- even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$

  e.g. x^2, $\cos x$, $|x|$

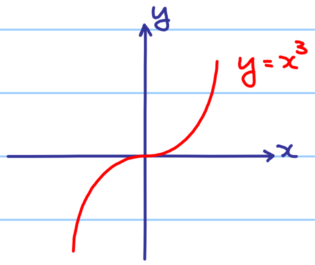
  property : the graph is symmetric along y-axis .



- odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$

  e.g. x^3 , $\sin x$

  property : the graph is symmetric about the origin

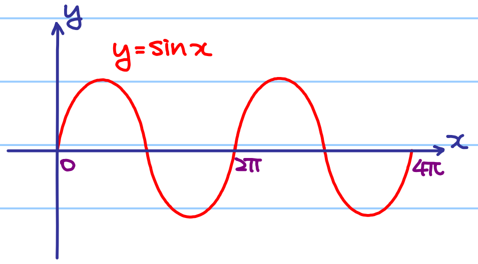


- periodic if there exists $T>0$ such that $f(x)=f(x+T)$ for all $x\in\mathbb{R}$

  If $T>0$ is the least positive real number with the above property, T is called the period.

  e.g. $\sin x$, $\cos x$, $\tan x$

  property: the graph is repeating again and again



period of $\sin x, \cos x = 2\pi$

period of $\tan x = \pi$

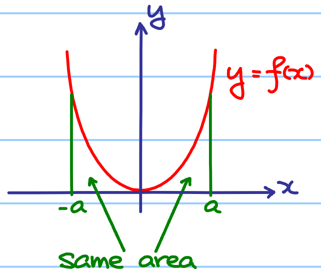
Example 9.4.3

Suppose f is an even function and $a>0$, prove that $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$\overset{?}{\|}$

$$\int_0^a f(x)\,dx$$



$\int_{-a}^{0} f(x)\,dx$ Let $y=-x$

$dy=-dx$

$= \int_{a}^{0} -f(y)\,dy$ When $x=0, y=0$

$f(-y)=f(y)$

$= \int_0^a f(y)\,dy$ $x=-a, y=a$

$= \int_0^a f(x)\,dx$ (dummy variable)

For example, $|x|$ is an even function.

$\int_{-4}^{4} |x|\,dx = 2\int_0^4 |x|\,dx = 2\int_0^4 x\,dx = 2\left[\frac{x^2}{2}\right]_0^4 = 16$

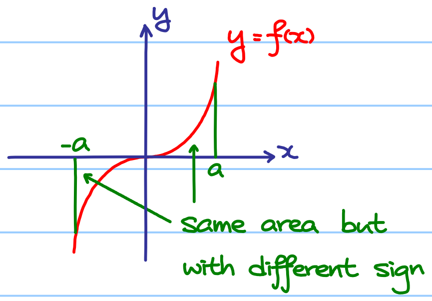
Example 9.4.4

Suppose f is an odd function and $a>0$, prove that $\int_{-a}^{a} f(x)\,dx = 0$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$\| \, ?$

$-\int_{0}^{a} f(x)\,dx$



$\int_{-a}^{0} f(x)\,dx$ Let $y=-x$

$= \int_{a}^{0} -f(-y)\,dy$ $dy = -dx$

$f(-y) = -f(y)$ ↓

$= \int_{0}^{a} -f(y)\,dy$ When $x=0,\ y=0$

$x=-a,\ y=a$

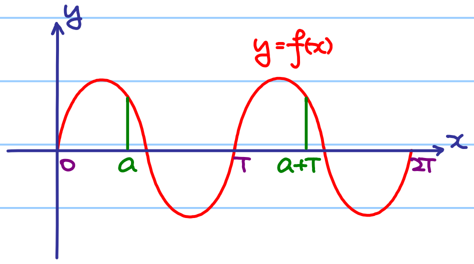
$= -\int_{0}^{a} f(x)\,dx$ (dummy variable)

For example, $\sin^{1001}x$ is an odd function, so $\int_{-\pi}^{\pi} \sin^{1001}x\,dx = 0$.

Example 9.4.5

Suppose f is a periodic function with period $T>0$ and $a \in \mathbb{R}$,

prove that $\int_{a}^{a+T} f(x)\,dx = \int_{0}^{T} f(x)\,dx$



$$\int_{a}^{a+T} f(x)\,dx = \int_{a}^{T} f(x)\,dx + \int_{T}^{a+T} f(x)\,dx$$

$\| \, ?$

$\int_{0}^{a} f(x)\,dx$

$\int_{T}^{a+T} f(x)\,dx$ Let $y = x-T$

$= \int_{0}^{a} f(y+T)\,dy$ $dy = dx$

$f(y+T) = f(y)$ ↓

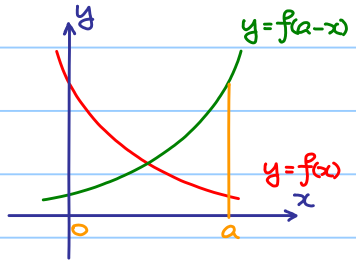
$= \int_{0}^{a} f(y)\,dy$ When $x=T,\ y=0$

$x=a+T,\ y=a$

$= \int_{0}^{a} f(x)\,dx$ (dummy variable)

Example 9.4.6

Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$



$\int_0^a f(a-x)\,dx$

Let $y = a-x$

$dy = -dx$

$= \int_a^0 -f(y)\,dy$

When $x=0,\ y=a$

$x=a,\ y=0$

$= \int_0^a f(y)\,dy$

$= \int_0^a f(x)\,dx$ (dummy variable)

9.5 Definite Integration Using Integration by Parts

Theorem 9.5.1

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\frac{du}{dx}\,dx$$

Example 9.5.1

Evaluate $\int_1^e x\ln x\,dx$

$$\int_1^e x\ln x\,dx = \int_1^e \ln x\ d\left(\frac{x^2}{2}\right)$$

$$= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}\,d\ln x$$

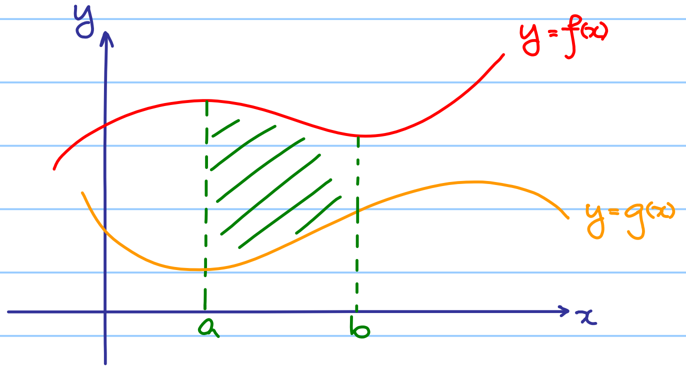
$$= \left(\frac{e^2}{2}\ln e - \cancelto{0}{\frac{1}{2}\ln 1}\right) - \int_1^e \frac{x}{2}\,dx$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e$$

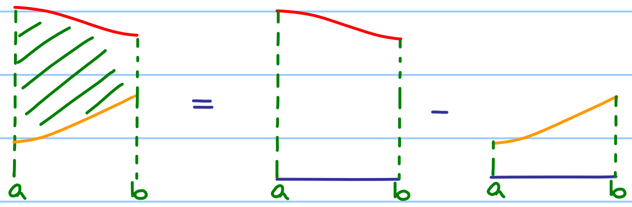
$$= \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

9.6 Area Between Curves



Area of shaded region = $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$



Example 9.6.1

Find the area bounded by $y = x^2$ and $y = x^3$.

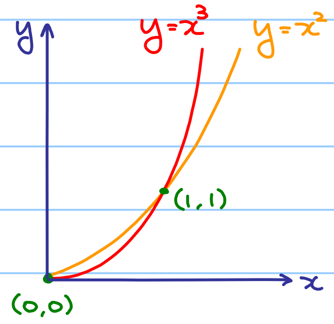
Step 1 : Solve $\begin{cases} y = x^2 \\ y = x^3 \end{cases}$

$$x^3 = x^2$$

$$x^2(x-1) = 0$$

$$x = 0 \quad \text{or} \quad 1$$

(Remark : No need to solve y)



Step 2 : Note when $0 \le x \le 1$, $x^3 \le x^2$

$$\text{Area} = \int_0^1 x^2 - x^3\,dx$$

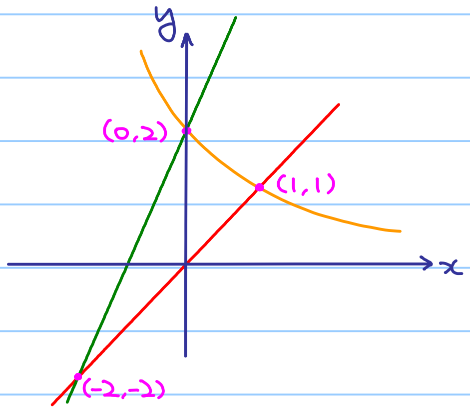
$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{12}$$

Example 9.6.2

Find the area bounded by

$y=f(x)=x$, $y=g(x)=\frac{2}{x+1}$ and $y=h(x)=2x+2$



Area $=\int_{-2}^{0} h(x)-f(x)\,dx+\int_{0}^{1} g(x)-f(x)\,dx$

Exercise :

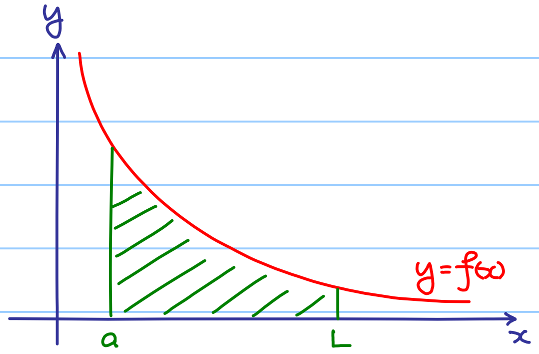
$=2+(-\frac{1}{2}+\ln 4)$

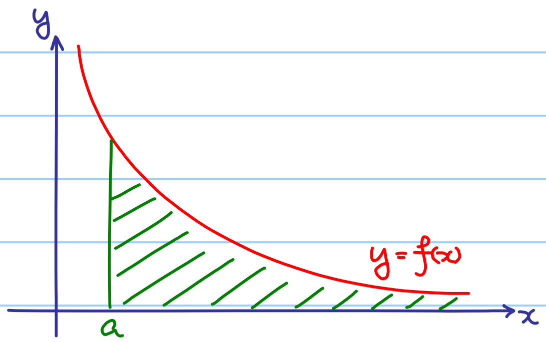
$=\frac{3}{2}+\ln 4$

9.7 Improper Integrals

Question : Find the area of the unbounded region ?

Idea :



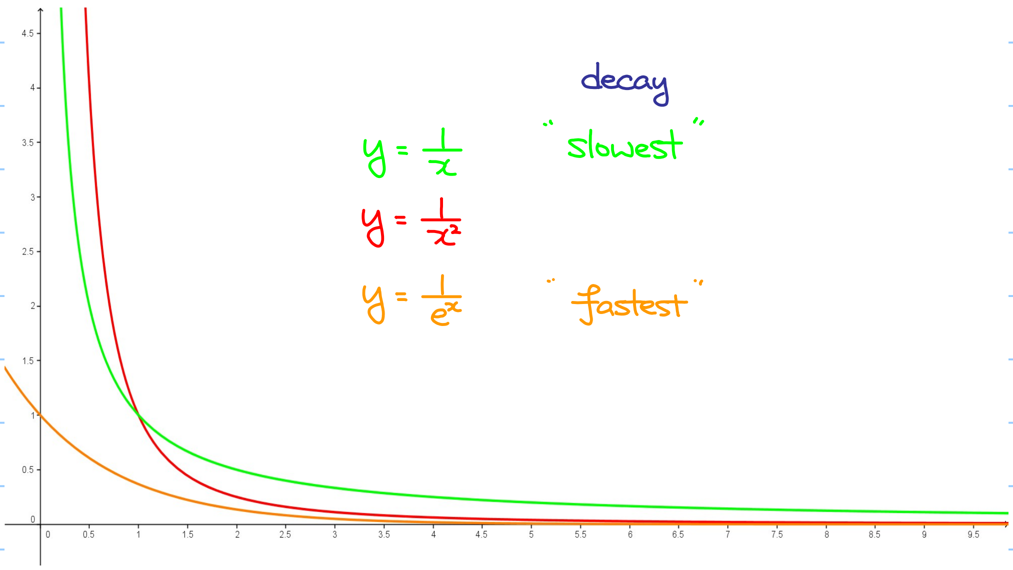


$\int_a^L f(x)dx$ $\rightsquigarrow$ Area of the unbounded region

$=\lim_{L\to+\infty}\int_a^L f(x)dx$ (if it exists)

We denote it by $\int_a^{+\infty} f(x)\,dx$

Example 9.7.1



① $\lim_{L\to+\infty} \int_1^L \frac{1}{x} dx = \lim_{L\to+\infty} [\ln x]_1^L = \lim_{L\to+\infty} \ln L = +\infty$ (i.e. limit does NOT exist)

② $\lim_{L\to+\infty} \int_1^L \frac{1}{x^2} dx = \lim_{L\to+\infty} [-\frac{1}{x}]_1^L = \lim_{L\to+\infty} 1 - \frac{1}{L} = 1$

③ $\lim_{L\to+\infty} \int_1^L \frac{1}{e^x} dx = \lim_{L\to+\infty} [-\frac{1}{e^x}]_1^L = \lim_{L\to+\infty} -\frac{1}{e^L} + \frac{1}{e} = \frac{1}{e}$

Observation : $\lim_{x\to+\infty} f(x) = 0$ does NOT guarantee $\lim_{L\to+\infty} \int_a^L f(x) dx$ exists.

Example 9.7.2

Find $\int_0^{+\infty} \frac{1}{(x+1)(3x+2)} dx$

Note : $(x+1)(3x+2)$ is a polynomial of degree 2.

$\frac{1}{(x+1)(3x+2)}$ decays as "fast" as $\frac{1}{x^2}$.

$$\lim_{L\to+\infty} \int_0^L \frac{1}{(x+1)(3x+2)} dx = \lim_{L\to+\infty} \int_0^L \frac{-1}{x+1} + \frac{3}{3x+2} dx$$
$$= \lim_{L\to+\infty} [\ \ln|x+1| + \ln|3x+2|\ ]_0^L$$
$$= \lim_{L\to+\infty} \ln\left|\frac{3L+2}{L+1}\right| - \ln 2$$
$$= \ln 3 - \ln 2$$

Example 9.7.3

Find $\int_0^{+\infty} xe^{-2x}dx$

$$\lim_{L\to+\infty}\int_0^L xe^{-2x}dx$$

$$=\lim_{L\to+\infty}\int_0^L x\,d(-\frac{1}{2}e^{-2x})$$

$$=\lim_{L\to+\infty}[-\frac{1}{2}xe^{-2x}]_0^L-\int_0^L-\frac{1}{2}e^{-2x}dx$$

$$=\lim_{L\to+\infty}[-\frac{1}{2}xe^{-2x}]_0^L+[-\frac{1}{4}e^{-2x}]_0^L$$

tend to 0 when L→+∞

$$=\lim_{L\to+\infty}-\frac{1}{2}\overset{\downarrow}{Le^{-2L}}-\frac{1}{4}\overset{\downarrow}{e^{-2L}}+\frac{1}{4}$$

$$=\frac{1}{4}$$

9.8 Inequalities Involving Integrals

Example 9.8.1

Define $a_n=\int_0^1\frac{x^n}{1+x^2}dx$ for $n\in\mathbb{N}$, show that $\frac{1}{2(n+1)}\leq a_n\leq\frac{1}{n+1}$.

Hence, find $\lim_{n\to\infty}a_n$.

Note : For $0\leq x\leq 1$, $\quad 1\leq 1+x^2\leq 2$

$$\frac{1}{2}\leq\frac{1}{1+x^2}\leq 1$$

$$\frac{x^n}{2}\leq\frac{x^n}{1+x^2}\leq x^n$$

$$\int_0^1\frac{x^n}{2}dx\leq\int_0^1\frac{x^n}{1+x^2}dx\leq\int_0^1 x^n dx$$

$$\frac{1}{2(n+1)}\leq\int_0^1\frac{x^n}{1+x^2}dx\leq\frac{1}{n+1}$$

$$\because \lim_{n\to\infty}\frac{1}{2(n+1)}=\lim_{n\to\infty}\frac{1}{n+1}=0$$

$\therefore$ By sandwich theorem, $\lim_{n\to\infty}\int_0^1\frac{x^n}{1+x^2}dx=0$

Example 9.8.2

Let $I_n = \frac{1}{n}\int_0^1 \frac{\sin nx}{1+x^2}\,dx$, for $n \in \mathbb{N}$, prove that $|I_n| \leq \frac{\pi}{4n}$.

Hence, deduce $\lim\limits_{n\to\infty} I_n = 0$.

$$|I_n| = \frac{1}{n}\left|\int_0^1 \frac{\sin nx}{1+x^2}dx\right|$$

$$\leq \frac{1}{n}\int_0^1 \left|\frac{\sin nx}{1+x^2}\right| dx$$

$$\leq \frac{1}{n}\int_0^1 \frac{1}{1+x^2}dx$$

$$= \frac{1}{n}[\tan^{-1}x]_0^1$$

$$= \frac{\pi}{4n}$$

$0 \leq |I_n| \leq \frac{\pi}{4n}$ and $\lim\limits_{n\to\infty} 0 = \lim\limits_{n\to\infty} \frac{\pi}{4n} = 0$

$\therefore$ By sandwich theorem, $\lim\limits_{n\to\infty} |I_n| = 0$ and so $\lim\limits_{n\to\infty} I_n = 0$.

Example 9.8.3

Let $I_n = \int_0^1 e^t t^n dt$ where n is nonnegative integer.

a) Prove that $I_n = e - nI_{n-1}$ for $n \geq 1$.

Hence, deduce that $I_n = (-1)^{n+1} n! + e\sum\limits_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!}$.

b) Show that $\frac{1}{n+1} \leq I_n \leq \int_0^1 et^n dt < \frac{e}{n}$ for all $n \geq 1$.

c) Hence, prove that e is an irrational number.

a) For $n \geq 1$, $I_n = \int_0^1 e^t t^n dt$

$$= \int_0^1 t^n de^t$$

$$= [t^n e^t]_0^1 - \int_0^1 e^t dt^n$$

$$= e - \int_0^1 ne^t t^{n-1} dt$$

$$= e - nI_{n-1}$$

$I_n = e - nI_{n-1}$

$= e - n[e-(n-1)I_{n-2}]$

$= e - ne + n(n-1)I_{n-2}$

$= e - ne + n(n-1)[e-(n-2)I_{n-3}]$

$= e - ne + n(n-1)e - n(n-1)(n-2)I_{n-3}$

$\vdots$

$= e - ne + n(n-1)e - \cdots + (-1)^{n-1}n(n-1)(n-2)\cdots 2e + (-1)^n n(n-1)(n-2)\cdots 2\cdot 1\cdot I_0$ $\qquad I_0 = \int_0^1 e^t dt$

$= e - ne + n(n-1)e - \cdots + (-1)^{n-1}n(n-1)(n-2)\cdots 2e + (-1)^n n(n-1)(n-2)\cdots 2\cdot 1\cdot (e-1)$ $\qquad = [e^t]_0^1$

$= e - ne + n(n-1)e - \cdots + (-1)^{n-1}n(n-1)(n-2)\cdots 2e + (-1)^n n(n-1)(n-2)\cdots 2\cdot 1\cdot e$ $\qquad = e-1$

$+(-1)^{n+1}n(n-1)(n-2)\cdots 2\cdot 1$

$= (-1)^{n+1}n! + e\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!}$

b) Note : For $0 \le t \le 1$, $\qquad 1 \le e^t \le e$

$$t^n \le e^t t^n \le et^n$$

$$\int_0^1 t^n dt \le \underset{\substack{\shortparallel \\ I_n}}{\int_0^1 e^t t^n dt} \le \int_0^1 et^n dt$$

$\int_0^1 t^n dt = [\frac{1}{n+1}t^{n+1}]_0^1 = \frac{1}{n+1}$

$\int_0^1 et^n dt = e\int_0^1 t^n dt = \frac{e}{n+1} < \frac{e}{n}$

$\therefore \frac{1}{n+1} \le I_n \le \int_0^1 et^n dt < \frac{e}{n}$

c) From (a) and (b),

$$\frac{1}{n+1} \le I_n < \frac{e}{n}$$

$$\frac{1}{n+1} \le (-1)^{n+1}n! + e\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!} < \frac{e}{n}$$

$$\frac{1}{e(n+1)} \le \frac{(-1)^{n+1}n!}{e} + \sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!} < \frac{1}{n}$$

If e is rational, i.e. $e = \frac{p}{q}$ where $p, q \in \mathbb{N}$ (Note $e > 0$)

when we consider $n = p$,

then $\frac{(-1)^{n+1}n!}{e} + \sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!}$ is an integer which is impossible as $0 < \frac{1}{e(n+1)}$ and $\frac{1}{n} \le 1$!

$\therefore$ e is irrational.